# Lecture 9 — Numerical experimentation

## Numerical versus exact orbits

- I must not continue with a synthesis of analysis and numerics without addressing the question of what relationship exists between them.
- There are some general results about this question, known as *shadowing theorems* — basically, numerically computed orbits are shadowed by exact orbits, in the sense that there are always exact orbits close to computed ones, even over long time intervals.
- More precisely, if $\bar{x}_0, \cdots, \bar{x}_k$ is numerically computed, with typical error at each step in the order of $\epsilon$, then there is an exact orbit $z_0, \cdots, z_k$ for which $|\bar{x}_i - z_i| \approx \epsilon$ for $i = 0, \cdots, k$.
- Note that $\bar{x}_0 \neq z_0$: the shadow really is a different orbit.
- For example, our earlier unsuccessful attempt to reproduce an unstable period 2 orbit numerically produced something which was close to an actual *chaotic orbit.* It was *not* close to the period 2 orbit we set out to find.

## Accumulation of errors for the tent map

- Suppose that $x_k$ is an orbit of the tent map with $t = 1$, that it, it exactly satisfies

$$x_{k+1} = \begin{cases} 2x_k, & (x_k \leq 1/2 \\ 2 - 2x_k, & (x_k \geq 1/2) \end{cases}$$

- Let's represent our attempt to calculate this numerically by the sequence $\bar{x}_k$. The numerics begins with

$$\bar{x}_0 = x_0 + \epsilon_0,$$

where $\epsilon_0$ is the error in representing $x_0$ numerically.

- After the first iteration

$$\bar{x}_1 = f(\bar{x}_0) + \epsilon_1,$$

where $\epsilon_1$ is the error in calculating $f(\bar{x}_0)$.

- The difference between exact and numerical values has now grown to $\bar{x}_1 - x_1 = \pm 2\epsilon_0 + \epsilon_1$, and the ± sign is determined according as $\bar{x}_0 < 1/2$ or $\bar{x}_0 > 1/2$.

- After $k$ steps we have

$$\bar{x}_k - x_k = \pm 2^k \epsilon_0 \pm 2^{k-1}\epsilon_1 \pm \cdots + \epsilon_k,$$

- We can see the difference growing rapidly until there is no apparent relationship between the two sequences — sensitive dependence on initial conditions.

### An orbit in the shadow

- To keep track of the source of errors, I have using the symbols $\epsilon_i$ above. All these errors have the same numerical origin, and they will all be measured in units of some small quantity $\epsilon$ related to the computational process.
- $f_k$ is a continuous function, so we may examine other exact orbits commencing from values $z_0$ near $x_0$, and look for a value for which each $z_k$ is close to the computed $\bar{x}_k$.
- As a first step, if $\bar{x}_1 \neq x_1$, then it is easy to find a nearby $z_0$ such that $\bar{x}_1 = z_1$. Furthermore, because past errors double at each iteration, the distance between $z_0$ and $x_0$ will be only about $\epsilon/2$.
- At the next iteration, $\bar{x}_2$ will differ from both $x_2$ and $z_2$, although for the latter the difference will only be because of the error introduced at the second step.
- At this point, replace $z_0$ ($z_0^{\text{old}}$) by $z_0^{\text{new}}$, chosen so that $\bar{x}_2 = z_2^{\text{new}}$ (which of course means that we no longer have agreement at the first iteration). To do this we must shift the second iteration by an amount of the order $\epsilon$.
- Once more the magnification factor comes to the rescue: the shift from $z_1^{\text{old}}$ to $z_1^{\text{new}}$ is in the order of $\epsilon/2$, and the shift from $z_0^{\text{old}}$ to $z_0^{\text{new}}$ is in the order of $\epsilon/4$.
- Now the process of finding the shadowed orbit is clear. We repeatedly refine the choice of $z_0$, applying at the $k$th step the condition

$$z_k^{\text{new}} - \bar{x}_k = 0.$$

- This will involve shifting the previous $z_{k-1}^{\text{old}}$ by about $\epsilon/2$, the previous $z_{k-2}^{\text{old}}$ by about $\epsilon/4$, right back to a shift in the previous $z_0^{\text{old}}$ by about $\epsilon/2^k$.

- All of these shifts are cumulative, but in a controlled fashion, since

$$1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^k} < 2.$$

- So, for any $k$, we may find an exact orbit $z_0, \cdots, z_k$ which is close to the numerically computed orbit $\bar{x}_0, \cdots, \bar{x}_k$, at *each step.*

### Final state diagrams

- The computational limit imposed in solving equation (8.1) necessitates observation using numerical iteration. (This is all the more so for higher-dimensional maps and unavoidable in experimental situations.)
- Such observations are displayed as "final states". What do they approximate?

### Forward limit sets

- We would like to observe the limiting set of points to which the system is attracted.
- In practice one can only make computations of orbits which are numerical shadows of actual orbits. It is important to be clear about the structure we are attempting to glimpse before proceeding.
- **Forward limit set:** The forward limit set of an orbit is the set of points which the orbit approaches arbitrarily closely infinitely often.
- This definition is quite subtle, containing two references to the infinite.

(1) Arbitrarily closely means that for any distance $\delta$, no matter how small, and for any point $\xi$ in the forward limit set, there are points on the orbit which satisfy $|x_k - \xi| < \delta$.

(2) Infinitely often means that there are always recurrences of this event: for any $N$ there are always values of $k > N$ for which $|x_k - \xi| < \delta$.

- A stable periodic orbit of a map is itself the forward limit set of all orbits within its basin of attraction $B$. There are an uncountable number of points in $B$; with the exception of the fixed point itself, none of them are in the forward limit set.
- For chaotic orbits, the situation is far more complex, and the forward limit set can contain an uncountably infinite number of points. Such sets are usually called *chaotic attractors.*

## Numerical approximation

- The problem for numerics is that the map should be iterated for an infinitely long time before any samples are taken — in practice one first iterates the map a few hundred or thousand times, then takes a sample of a similar number of iterations.
- The sample itself is put into *bins* which are intervals whose size is the $x$ (vertical) pixel resolution, while the parameter range is traversed in increments whose size is the (horizontal) pixel resolution.
- I refer to such diagrams as *final state diagrams*.
- As an example, final state diagrams for the logistic map exhibit quite clearly the birth and death of the period 2 orbit already investigated. (Overhead_9_1 & 9_2)
- It also shows that the period doubling repeats itself, on an ever finer scale, up to the resolution of the picture.
- It does not show is that periodic orbits do not really die, they simply become unstable.
- Eventually there are an infinite number of such unstable orbits in the background, creating chaos.
- Final state diagrams are *fractal* structures in chaotic regions of the dynamical system, so one can never see all the detail, regardless of how closely one looks.

## Cubic #3 map

- This is a *non-unimodal* map of the interval $[-1, 1]$, defined by the function

$$f(x) = x(1 - p + px^2), \qquad 0 \leq p \leq 4$$

- Here are four overheads for this map.
- The first two are final state diagrams with initial conditions at $\pm 0.499999$. (Overhead_9_3 & 9_4) Evidently there are *two* competing attractors!
- The second two are bifurcation diagrams, showing all orbits up to period 4 and period 48. The complexity is obvious. (Overhead_9_5 & 9_6)
- The bifurcation at $p = 2$ is *period doubling*, with $f'(x^*_\pm) = -1$. (Overhead_9_7) The bifurcation at $p = 3$ is *not period doubling*. In fact, $f_2'(x^*_\pm) = +1$. (Overhead_9_8)