Lecture 8 — Bifurcations

Source material: Chapter 3, pp 65–70, plus other material.

To reproduce overheads shown in lectures, download the corresponding files from the website and open them with "Chaos for Java"

- Bifurcation theory is concerned with the structure of fixed points and periodic orbits, stable or unstable. We have already seen two examples:

(1) The logistic map has two fixed points, given by

$$x_0^* = 0, \qquad x_1^* = 1 - 1/r.$$

Let's allow x be be any real number, then x_0^* is stable for $0 \le r < 1$ and unstable for $1 < r \le 4$, while x_1^* is unstable for $0 \le r < 1$ and stable for $1 < r < 3$.

(2) We already noted period doubling of the logistic map. At $r = 3$ a fixed point changes from stable to unstable; simultaneously a period 2 orbit is "born".

- A *bifurcation diagram* shows the fixed points and orbits, as functions of r, and their stability. It may only be constructed by solving the fixed point equations

$$x^* - f_n(x^*) = 0, \tag{8.1}$$

consulting the table of periodic orbits, then calculating values of the derivatives $f_n'(x^*)$ in each case to determine stability.

- The bifurcations of (1) and (2) above are shown. (Overheads 8_1 & 8_2)

(1) As r is increased, the two fixed points collide, whereupon they exchange their stability — a *transcritical bifurcation* — not the most common type of bifurcation.

(2) As r is increased, the fixed point x_1^* becomes unstable, and two new stable fixed points of f_2 are "born" as a period 2 orbit — a *period doubling bifurcation.*

- Three points emerge:

(1) There is a structural change at a critical parameter value.

(2) There is a kind of *conservation of stability*, related to the difference between the number of stable and unstable fixed points.

(3) By the time $r = 4$ there are a dense set of unstable periodic orbits. (Overhead_8_3)

- **Bifurcation:** A bifurcation is a change in the structure of the periodic orbits of a dynamical system as a system parameter varies continuously through a critical value.

The critical condition

- Bifurcation diagrams are produced by finding solutions of (8.1) as functions of r. For a selected value $\bar{r}$ of r, a corresponding solution $\bar{x}$ generally lies on a smooth curve $x^*(r)$ passing through the point $r = \bar{r}$, $x = \bar{x}$.
- For example, the fixed point x_1^* of the logistic map lies on the smooth curve $x_1^*(r) = (1 - 1/r)$, even as it changes its stability at $r = 3$.
- In contrast, x_1^* is also a fixed point of the second composition map f_2, but two new fixed points branch off at $r = 3$. Evidently, solutions of $\phi_2(x) = 0$ are not smooth in this case.
- What makes the curve smooth in the normal case? For smooth functions, we may use the local approximation

$$\phi_n(x; r) \approx A(r - \bar{r}) + B(x - \bar{x}) = 0, \tag{8.2}$$

where

$$A = [\partial\phi_n/\partial r]_{\bar{x},\bar{r}}\,, \qquad B = [\partial\phi_n/\partial x]_{\bar{x},\bar{r}}\,, \tag{8.3}$$

- This gives the slope of the tangent to the curve:

$$dx^*/dr = -A/B.$$

- For smooth function of r and x, the tangent is continuous, and the curve smooth, except in the singular case that $B = 0$, which is equivalent to $f_n'(x^*) = +1$.
- The vital conclusion is that curves $x^*(r)$ are smooth except for those critical points where $f_n'(x^*) = +1$.
- What about the other condition for loss of stability, namely $f_n'(x^*) = -1$? The fixed points of f_n, and hence the period n orbits, do not change their structure as curves in the r-x plane. However, fixed points of f_n are also fixed points of f_{2n}, with the property

that

$$f'_{2n}(x^*) = f'_n(x^*)^2 = +1.$$

- This is the origin of period doubling.

Unfolding a bifurcation

- The term "unfolding" comes from *catastrophe theory*. It is a standard result of this theory that there are only a small number of possibilities, particularly in the case that there is only one parameter, as here.
- Let's begin by reconsidering (8.2) using the more convenient notation that

$$x - x^* = \delta x, \qquad r - r^* = \delta r.$$

Then it takes the simple form

$$\phi_n(x; r) \approx A\delta r + B\delta x = 0,$$

which is a first order equation in δx whose unique solution was already given in (8.3). *Locally*, the solution is a straight line; this is the hallmark of a smooth curve — if you zoom in sufficiently, a curve just looks like a straight line!

Higher order approximation

- The coincidence that

$$\phi_n(x^*; r^*) = 0, \qquad \phi'_n(x^*; r^*) = 0, \tag{8.4}$$

is restricted to that special point in the $x - r$ plane (in general).

- If we move away from that point, then the lowest order approximations are

$$\phi_n(x; r) \approx A\delta r,$$

$$\phi'_n(x; r) \approx B\delta r,$$

$$\phi''_n(x; r) \approx \phi''_n(x^*; r^*) = 2C,$$

where A and B are values of the respective partial derivatives, taken at the point (x^*, r^*), namely

$$A = [\partial\phi_n/\partial r]_{x^*,r^*}, \qquad B = [\partial\phi'_n/\partial r]_{x^*,r^*}.$$

- The fixed point equation, to this approximation, is

$$A\delta r + B\delta r\delta x + C\delta x^2 = 0. \tag{8.5}$$

- I call it a *singular* quadratic equation. "Singular" because the leading terms are zero when $r = r^*$. For the present purposes I assume that $C \neq 0$, which would be yet another coincidence.

- Assuming that A and B are not zero, the solution is

$$\delta x \approx \frac{-B\delta r \pm \sqrt{-4AC\delta r}}{2A}. \tag{8.6}$$

- Once again, we see division by a coefficient — the very nature of the solutions will depend on whether or not $A = 0$, although nothing hinges on whether or not $B = 0$.

Transcritical bifurcation

- The case that $A = 0$ is obviously very special, since it constitutes yet a *third* coincidence after those of equations (8.4). However, I treat it right away because we have already observed such a bifurcation, for the logistic map at $r = 1$. Using results given previously, we have

$$A = \left[\frac{\partial\big(x - rx(1-x)\big)}{\partial r}\right]_{0,1} = 0$$
$$B = \big[\partial\big(1 - r(1-2x)\big)/\partial r\big]_{0,1} = 0.$$

- Let's revisit (8.5). Setting $A = 0$, the fixed point equation simplifies to

$$B\delta r\delta x + C\delta x^2 = 0.$$

The solutions are

$$x_0^* \approx x^*(r^*), \qquad (\delta x = 0),$$
$$x_1^* \approx x^*(r^*) - B\delta r/C, \qquad (\delta x = -B\delta r/C).$$

- How about stability? Our approximation gives

$$f_n'(x;r) \approx 1 - B\delta r - 2C\delta x. \tag{8.7}$$

so

$$f_n'(x_0^*; r) \approx 1 - B\delta r.$$

- Clearly this changes stability as r passes through the critical value.
- Now we compute, to this approximation, the derivative for the other fixed point. Substituting into (8.7) gives

$$\begin{aligned} f_n'(x_1^*, r) &\approx 1 - B\delta r - 2C(-B\delta r/C) \\ &= 1 + B\delta r. \end{aligned}$$

- This shows that the two fixed points really do *exchange* their stability at a transcritical bifurcation, just as we observed in the logistic map.
- Overheads_8_4 to 8_6 show some graphs for the logistic map.

Tangent bifurcation

- The more usual case in solving (8.5) is that $A \neq 0$, so that the solutions are given by (8.6). In this case there are no solutions on one side of $r = r^*$, two solutions on the other side.
- We have already seen such a bifurcation for the Cubic #1 map defined in (2.3) — see (3.3) for the formula for the fixed points. Unlike the approximate treatment above, the fixed-point equation may be solved *exactly* in this case.
- It is easy to calculate the second derivative of the Cubic #1 map:

$$f''(x) = -\frac{81}{8} r x,$$

  which is clearly non-zero since the fixed points $x_\pm^* > 0$. This is the condition required for a tangent bifurcation.
- Overheads_8_7 to 8_9 show some graphs for the logistic map.
- I shall revisit the tangent bifurcation in a later lecture.