Lecture 7 — Lyapunov Exponents

Source material: Chapter 2, pp 51–54

To reproduce overheads shown in lectures, download the corresponding files from the website and open them with "Chaos for Java"

- Stability of orbits of a map f is measured by the derivative f', whether or not the orbit is periodic. That is the subject of this lecture.

Periodic orbits

- We know that, if x_0^* is a point on a periodic orbit, and if we start the system from a nearby point x_0 whose distance from x_0^* is δ_0, then after one iteration the distance between the two is approximated by

$$\delta_1 \approx |f'(x_0^*)|\delta_0 = M_0 \cdot \delta_0.$$

  M_0 is the "magnification factor" for this step.

- At the next step

$$\delta_2 \approx |f'(x_1^*)|\delta_1 = M_1\delta_1 \approx M_1|f'(x_0^*)|\delta_0 = M_0M_1 \cdot \delta_0,$$

  where $M_1 = |f'(x_1^*)|$ is the magnification factor for the second iteration.

- For a period-n orbit, the total magnification factor over the n steps is the product

$$M_0M_1 \cdots M_{n-1}.$$

Long-term average

- Since this is an accumulation of magnification factors, it makes sense to consider an average magnification factor.
- If we take logarithms, then the products becomes a sum:

$$\ln(M_0M_1 \cdots M_{n-1}) = \ln M_0 + \cdots + \ln M_{n-1} = \ln|f'(x_0^*)| + \cdots + \ln|f'(x_{n-1}^*)|.$$

- The condition for stability of a period-n orbit is that the total magnification factor should be less than 1. If it is greater than 1 the orbit is unstable. After taking logarithms, the condition for stability is that the logarithm of the magnification factor is negative.

- Using logarithms, it is easy to define an average

$$L = \frac{1}{n}\left(\ln|f'(x_0^*)| + \cdots + \ln|f'(x_{n-1}^*)|\right),$$

which tells us about stability of the orbit: stable if $L < 0$ and unstable if $L > 0$.

- Now consider the long-term average,

$$L = \lim_{k\to\infty}\frac{1}{k}\sum_{i=0}^{k-1}\ln|f'(x_i^*)|.$$

For a period n orbit the average involves only the n quatities $|f'(x_i^*)|$, each multiplied by the number of times m_i the orbit visits x_i^* in the course of k iterations. The ratios m_i/k approach the limit $1/n$ for large k, showing that L is defined in this case.

- **Lyapunov exponent of one-dimensional map:** For a given initial point x_0, the Lyapunov exponent $L(x_0)$ of a map f is given by the formula

$$L(x_0) = \lim_{k\to\infty}\frac{1}{k}\sum_{i=0}^{k-1}\ln|f'(x_i)|,$$

provided the limit exists. In the case that any of the derivatives in the calculation is zero, we write $L(x_0) = -\infty$. It is clear from our discussion that the Lyapounov exponent of a periodic orbit is a property of the orbit as a whole:

$$L(x_0^*) = L(x_1^*)\cdots = L(x_{n-1}^*) = \frac{1}{n}\sum_{k=0}^{n-1}\ln|f'(x_k^*)| = \frac{1}{n}\ln|f_n'(x_k^*)|.$$

Lyapunov exponents for basins of attraction

- Suppose that we compute $L(x_0)$ for an eventually periodic orbit. After the first m iterations, we reach a point x_m which is actually on a periodic orbit. Then

$$L(x_0) = \lim_{k\to\infty}\frac{1}{k}\left(\sum_{i=0}^{m-1}\ln|f'(x_i)| + \sum_{i=m}^{k-1}\ln|f'(x_i)|\right).$$

- In the limit of large k, the second term dominates and we recover the Lyapunov exponent of the periodic orbit itself, $L(x_i^*)$.

- We can go a step further: if x_0 is within the basin of attraction of a periodic orbit, then the Lyapunov exponent will be *exactly the same* as that of the periodic orbit itself.

- To show this, we must demonstrate that for any small number ϵ, no matter how small, the difference between $L(x_0)$ and $L(x_0^*)$ is less than ϵ. Now the orbit is approaching the periodic attractor as k increases, so the differences between $f'(x_k)$ and $f'(x_k^*)$ are converging to zero. From this it is not difficult to see the desired result.

Chaos in bounded maps

- **Chaotic orbits:** A chaotic orbit of a bounded system is one which is not periodic or eventually periodic, and which has positive Lyapunov exponent.
- **Chaos:** A dynamical system will be said to be chaotic when it is in a regime with chaotic orbits.
- Actually showing that the Lyapunov exponent exists may be a difficult problem. Often numerical calculation is used as supporting evidence, but it is important to have a theoretical analysis of some simple cases to back this up.
- Positive Lyapunov exponent implies that the orbit never falls within the basin of attraction of any periodic orbits. In case the orbit is periodic, or eventually periodic, it must be unstable.
- Furthermore, given a chaotic orbit starting from position x_0, then for any distance, no matter how small, there are starting points within this distance of x_0 for which the two orbits eventually separate from each other on the overall scale of the system.
- Some definitions of chaos include a requirement that there be a dense set of unstable periodic orbits.

Tent map

- For the tent map, and for any orbit which does not map exactly to $x = 1/2$, where the derivative of f is not defined, we obviously have

$$L(x_0) = \ln 2t$$

This is in accordance with the divergence of orbits for $t > 1/2$ and convergence for $t < 1/2$, which we already noted.

- Since we have shown that the tent map for $t = 1$ has an infinite number of orbits which are not eventually periodic (every irrational number is on one), we have shown that it is a chaotic system with Lyapunov exponent $L(x_0) = \ln 2 \approx 0.693$ for almost all x_0.

Logistic map

- Recall that the fixed point $x_1^* = (r-1)/r$ is stable in the window $1 < r < 3$, so that $x_0 \to x^*$ for any x_0. It follows that

$$L(x_0) = \ln |f'(x^*)| = \ln |2-r|, \qquad (1 < r < 3).$$

  Notice that $L \to 0$ at $r \to 1$ and $r \to 3$, $L \to -\infty$ as $r \to 2$.

- For the fixed points, $L(x_0^*) = \ln |f'(x_0^*)| = \ln r$ for $0 \leq r \leq 4$, $L(x_1^*) = \ln |f'(x_1^*)| = \ln |2-r|$ for $1 < r \leq 4$.

- The 2-cycle is stable in the window $3 < r < 1 + \sqrt{6}$. Hence

$$L(x_0) = \frac{1}{2} \ln \left|4 + 2r - r^2\right|, \quad (x_0 \neq x_0^*, x_1^*), \quad (3 < r < 1 + \sqrt{6}).$$

  This time $L \to 0$ at $r \to 3$ and $r \to 1 + \sqrt{6}$, and $L \to -\infty$ as $r \to 1 + \sqrt{5} \approx 3.236$.

- For $r > 1 + \sqrt{6}$ there are two more points $x_\pm^*$ where the Lyapunov exponent has special values, $L(x_\pm^*) = 1/2 \ln \left|4 + 2r - r^2\right|$, $1 + \sqrt{6} < r \leq 4$.

- Attempts to check these special values numerically will fail unless an algorithm is used which can find and track unstable orbits.

Numerical estimation

- Numerical computations are made by iterating the map to try to achieve convergence to any attracting set of states, after which the average value of $|\ln f'(x_i)|$ is computed over some reasonably large sample. (Overheads 7_1 to 7_4 for logistic map)

- As with bifurcation diagrams, numerical results must be treated with caution. The use of a finite sample means that the exponent is only an estimate.

- However, there is also fractal structure, since there are an infinity of such windows. Hints of this are seen in the pictures which I have shown. Note that I used a larger sample size when constructing the closer view.

Using numerical evidence

- Spectral analysis can provide strong evidence of non-periodic behaviour. Given this, together with positive numerically estimated values of the Lyapunov exponent, one can be rather certain that a particular system is exhibiting chaos, even though a more theoretical demonstration is not feasible.
- As an example which is not completely trivial, consider the picture shown here, for the logistic map with $r = 3.9615$. The spectrum shows clear peaks at frequencies of 1/4 and 1/2, suggesting the possibility of a period-4 orbit. (Overhead 7_5)
- The sample size, and the time allowed for the system to reach a stable situation, are both large, despite which there is a significant amount of "noise". Nor is the noise caused by a bad choice of divisor: 2000 is divisible by 4.
- The Lyapunov exponent is positive ≈ 0.128, so the system is chaotic.
- I leave it to you to investigate precisely what is going on here. However, here is a clue. (Overhead 7_6)