Lecture 6 — Fourier Analysis

Source material: Chapter 2, pp 44–48

To reproduce overheads shown in lectures, download the corresponding files from the website and open them with "Chaos for Java"

Origins

- Joseph Fourier introduced the technique of representing a function as a linear combination, or superposition, of trigonometric functions. It applies both to continuous functions and to discrete ones.
- If a signal is periodic then the spectrum contains the basic frequency plus integer multiples thereof — the "harmonics". This is a familiar everyday idea, for example, in acoustics.
- The discrete case is appropriate for discrete dynamical systems. The theoretical basis, is probably unfamiliar to you, even if you have studied the topics of Fourier series or integrals elsewhere.
- It is easy to understand the meaning of spectral analysis, and what it tells us in the "frequency domain". Since actual calculations are always done using a computer package I shall concentrate on meaning and interpretation rather than theory.

Simple examples

- First, consider a period-2 orbit, with $x_k = x_+^*$ for even k, $x_k = x_-^*$ for odd k. Since $\sin \pi k = +1$ for even k and -1 for odd k, it is obvious that we can represent it using just two cosine functions:
$$x_k = a_0 \cos(0) + a_{1/2} \cos\left(2\pi \cdot \frac{1}{2} \cdot k\right).$$
- Formulae for the amplitudes a_0, $a_{1/2}$ are
$$a_0 = \frac{1}{2}(x_+^* + x_-^*), \qquad a_{1/2} = \frac{1}{2}(x_+^* - x_-^*).$$
From this point of view, the data has a zero-frequency component of amplitude a_0 and an oscillating component of amplitude $a_{1/2}$, whose frequency is half the frequency of iteration.

- Period 4 is almost as simple. This time we need to know the behaviour of $\cos \pi k/2$ and $\sin \pi k/2$. Starting with $k = 0$ they generate the sequences $1, 0, -1, 0, \cdots$ and $0, 1, 0, -1, \cdots$.
- From this it is easy to see that a period-4 orbit has the representation

$$x_k = a_0 + a_{1/4} \cos\left(2\pi \cdot \frac{1}{4} \cdot k\right) + b_{1/4} \sin\left(2\pi \cdot \frac{1}{4} \cdot k\right) + a_{1/2} \cos\left(2\pi \cdot \frac{1}{2} \cdot k\right),$$

where

$$\begin{aligned} a_0 &= \frac{1}{4}(x_0 + x_1 + x_2 + x_3), & a_{1/4} &= \frac{1}{2}(x_0 - x_2), \\ a_{1/2} &= \frac{1}{4}(x_0 - x_1 + x_2 - x_3), & b_{1/4} &= \frac{1}{2}(x_1 - x_3). \end{aligned}$$

- Again the interpretation is that of frequency components, this time at one-quarter and one-half of the iteration frequency.
- For a period-3 orbit

$$x_k = a_0 + a_{1/3} \cos\left(2\pi \cdot \frac{1}{3} \cdot k\right) + b_{1/3} \sin\left(2\pi \cdot \frac{1}{3} \cdot k\right),$$

with

$$\begin{aligned} a_0 &= \frac{1}{3}(x_0 + x_1 + x_2), \\ a_{1/3} &= \frac{2}{3}(x_0 + \cos\frac{2\pi}{3} x_1 + \cos\frac{4\pi}{3} x_2), \\ b_{1/3} &= \frac{2}{3}(x_0 + \sin\frac{2\pi}{3} x_1 + \sin\frac{4\pi}{3} x_2). \end{aligned}$$

- This follows after substituting the previous formulae for x_k into the right hand sides and then using the values of the trigonometric functions.

General representation

- There are general principles emerging from these simple examples.

(1) From an orbit of length N we expect to need the coefficients of about $N/2$ cosine functions and sine functions; equivalently, about $N/2$ amplitudes and phases.

(2) The observable frequency components will be in the range 0 to 1/2, since we cannot observe periodic components whose variation is faster than that.

- In general, if we have an orbit x_k, $n = 0, \cdots, N-1$, then there is a trigonometric representation

$$x_k = \sum_{0 \le m \le N/2} \left[a_{m/N} \cos\left(2\pi \cdot \frac{m}{N} \cdot k\right) + b_{m/N} \sin\left(2\pi \cdot \frac{m}{N} \cdot k\right) \right].$$

- From now on I assume that N is an even number, so that the maximum value of m is exactly $N/2$. This is no real restriction, since we shall want to make N large in typical applications, and there are other reasons to make N divisible by small numbers.
- The frequencies which may be detected in a finite-length sample (even N) are

$$\frac{k}{N} = \frac{0}{N}, \frac{1}{N}, \frac{2}{N}, \cdots, \frac{1}{2},$$

that is, a discrete set of frequencies from 0 to 1/2 in steps of $1/N$.

Amplitude and phase

- Terms with frequency m/N come in pairs.
- Using the trigonometric identity $\cos(\theta + \phi) = \cos\theta \cos\phi - \sin\theta \sin\phi$, they may always be combined to a single cosine (or sine) terms, with amplitude $A_{m/N} = \sqrt{a^2_{m/N} + b^2_{m/N}}$ and phase $\phi_{m/N}$, to give a simpler general formula

$$x_k = \sum_m A_{m/N} \cos\left(2\pi \cdot \frac{m}{N} \cdot k + \phi_{m/N}\right).$$

I spare the details: the point is that it is possible to use either form.

- I prefer to use the language of amplitude and phase, since they give independent information about the periodic components of an orbit. Moreover, we shall be interested solely in the amplitude when analysing orbits of dynamical systems.

Logarithmic amplitude scale

- In a typical calculation, there will be significant amplitudes which are still very small compared with the largest one — typically the ratios could be as small as $10^{-4} \sim 10^{-8}$.
- To see this on a graph, it is usual to display the amplitudes on a logarithmic scale.
- **Decibel:** If A is a positive number, then its decibel (dB) value is calculated as $20 \ln_{10} A$, where the logarithm is taken to the base of 10.

- Since $\ln_{10} 2 \approx 0.3$, each doubling or halving of an amplitude shows up as an increase or decrease by about 6dB on this scale.
- Lets look at some actual computations.
- First, iterating the logistic map with $r = 3.8285$, there is a stable period 3 attractor. (Overhead 6_1)
- This is seen more clearly by joining every third point, that is, by observing iteration of f_3 rather than f, then going out to 5000 iterations. (Overhead 6_2)
- Here is the spectrum obtained by sampling iterations #2001–5000, a sample of 3000 after to discarding 2000 iterations to allow time for convergence to the periodic orbit. (Overhead 6_3) The only significant amplitudes are at frequencies 0 and 1/3.

Fast Fourier Transform

- Calculating each of the $N/2$ coefficients $A_{m/N}$ and $\phi_{m/N}$ requires a sum over all N values x_k. This seems to imply a calculation whose time increases proportionally to N^2.
- The *Fast Fourier Transform* (*FFT*) algorithm is a particularly efficient algorithm, which requires an amount of computation which grows only in proportion to $N \ln N$.
- It is only effective if the number N is a multiple of only *small* prime numbers; for example $3840 = 2^8 \cdot 3 \cdot 5$ and $4000 = 2^5 \cdot 5^3$.
- As a consequence, *Chaos for Java* will allow you to choose a sample size of 33000 (11 is the largest factor) but not 34000 (17 is not allowed in the current version).

Choosing discard sizes

- Here is the spectrum for the first 100 iterations of the logistic map, starting from $x_0 = 0.1$, with $r = 2.99$, together with a picture of these iterations. (Overheads 6_4 & 6_5)

(1) Observe is that the zero-frequency component is dominant. This is in accordance with the fact that the map has a stable fixed point at this value of r.

(2) The iterations are quite obviously converging rather slowly, and via a pronounced period-2 transient. This is relected by an obvious peak in the spectrum at the corresponding frequency 1/2.

(3) All frequency components are significant ($\geq 0.5\%$) due to the fact that the state of the system is one of continual significant change during this initial phase.

- Now lets look at the spectrum with the same parameters, but after waiting some time for the iterations to approach the fixed point more closely.
- It is evident that we are approaching the simple situation where only A_0 and $A_{1/2}$ are non-zero, as we expect. (Overheads 6_6 & 6_7)
- The lesson is: if there is possibility of transient behaviour, while the system is approaching an attractor, then it pays to experiment with the discard size.
- This applies also to the estimation of Lyapunov exponents.

Choosing sample sizes

- For real efficiency the *FFT* requires that N have only small prime numbers as divisors. The most efficient choice is a power of 2. For example, if $N = 2^n$, the *FFT* requires n recursive steps, for each of which the number of calculations is proportional to N.
- However, there are other factors to be taken into account when selecting sample sizes. Consider the following two spectra, both taken for the logistic map with $r = 3.83$, at which value it has a stable period-3 orbit.
- One picture leaves no doubt about the period-3 behaviour, the other is not so clear. (Overheads 6_8 & 6_9)
- The difference between the two is that 3 divides 4050, but not 4000. The point is that, if the sample size is not an exact multiple of 3, then the data cannot be fitted exactly using only the frequencies 0 and 1/3.
- The lesson is clear. If, as the result of spectral analysis, exact periodic behaviour is suspected but not completely clear, then the question might be resolved by making sure that the sample size is divisible by that period.
- Here is another example with a period 7 orbit. (Overheads _10 & 6_11)