*Source material: Chapter 2, pp 30–35*

*To reproduce overheads shown in lectures, download the corresponding files from the website and open them with "Chaos for Java"*

## Lecture 5 — Periodic orbits

### Period doubling — logistic map

- What happens for the logistic map when $r > 3$? As with the tent map, both of its fixed points are unstable in this case, but the behaviour is totally different.
- Iterations starting from $x_0 = 0.7$, with $r = 3.4$, appear to converge to a rectangular path on the cobweb plot — a periodic orbit of period 2. (Overheads 5_1 & 5_2) Unlike the tent map, this orbit appears to be an attractor. I shall show that this is indeed the case.

### Formula for the orbit

- Let's find the fixed point of the second composition map $f_2(x) = f(f(x))$ of the logistic map. Substituting $f(x) = rx(1-x)$ for $x$ into $f(x)$ itself, we get

$$f_2(x) = r^2 x(1-x)(1 - rx + rx^2).$$

Here is the graph of $f_2$ when $r = 3.4$. (Overhead 5_3)

- To find the fixed points of $f_2$, we must solve the fourth order polynomial equation

$$\phi_2(x) = x - f_2(x) = 0.$$

- Fixed points of $f$ must be fixed points of $f_2$, therefore two factors of $\phi_2(x)$ must be $x$ and $(x - 1 + 1/r)$. Using this ,

$$\phi_2(x) = x(1 - r + rx)(1 + r - rx - r^2 x + r^2 x^2).$$

- The new fixed points are the roots of the quadratic factor. Calling them $x^*_\pm$,

$$x^*_\pm = \frac{1 + r \pm \sqrt{r^2 - 2r - 3}}{2r}.$$

- The quantity under the square root sign factors as $r^2 - 2r - 3 = (r+1)(r-3)$. For real solutions this gives the condition $r \geq 3$ for the existence of the new fixed points $x^*_\pm$.

## Stability of period 2 orbits

- Suppose that $x_0^*$, $x_1^*$ is a period 2 orbit of a map $f$ which is smooth, at least in two intervals $I_0$ and $I_1$ which contain the point $x_0^*$ and $x_1^*$ respectively.

(1) Suppose that the *orbit* is stable, so that a trajectory starting from any $x_0$ in $I_0$ converges to the periodic orbit.

- Then the sequence of values $x_0, x_2, x_4, \cdots$ converge to $x_0^*$, with the alternating values $x_1, x_3, x_5, \cdots$ converging to $x_1^*$. The actual orbit under consideration consists of two subsequences, each of which converges to one of the points $x_0^*, x_1^*$.
- This shows that both points on a stable period 2 orbit are stable fixed points of the second composition map $f_2$.

(2) Suppose that $x_0^*$ and $x_1^*$ are *stable fixed points of* $f_2$, with $x_1^* = f(x_0^*)$, $x_0^* = f(x_1^*)$.

- Choose an interval from which all orbits of $f_2$ converge to $x_0^*$. Label them as $x_0, x_2, x_4, \cdots$, and define odd-numbered points by $x_1 = f(x_0)$, $x_3 = f(x_2)$, $x_5 = f(x_4), \cdots$.
- Because $f_2(x_0) = x_2$, it follows that $x_2 = f(x_1)$, similarly $x_4 = f(x_3)$, $x_6 = f(x_5), \cdots$. The totality of points thus defined are a single orbit if $f$.
- Because the map $f$ is smooth, the odd-numbered points converge to $x_1^*$ and this orbit of $f$ converges to the period 2 orbit $x_0^*, x_1^*$.
- We needed something extra (smoothness) to make the argument work in this direction.

## Derivative test for period 2 orbit

- Now let's apply the derivative test to the pair $x_0^*$ and $x_1^*$. First we need the derivative.
- Recall the chain rule for a function of a function, $f(g(x))$:

$$\frac{df}{dx} = \frac{df}{dg}\frac{dg}{dx} = f'(g(x)) \cdot g'(x).$$

Here $f$ and $g$ are the same, and we get

$$f_2'(x) = f'(f(x)) \cdot f'(x). \tag{5.1}$$

- Now we must find the value of $f_2'$ at each of the two fixed points. The important fact is that $f(x_0^*) = x_1^*$, and conversely, so that

$$f_2'(x_0^*) = f'(x_1^*) \cdot f'(x_0^*) = f_2'(x_1^*).$$

- Important conclusion: stability is a property of the orbit; the derivative in the test involves $f'$ at each of the two points, once each and once only.
- A period 2 orbit $x_0^*$, $x_1^*$ is stable if $|f'(x_0^*)f'(x_1^*)| < 1$ and unstable if $|f'(x_0^*)f'(x_1^*)| > 1$.
- The derivative test applies to the period 2 orbit of the tent map (equations (4.1)) and shows that it is unstable:

$$f_2'(x_1^*) = f_2'(x_2^*) = f'(x_1^*)f'(x_2^*) = (2t)^2 > 1, \qquad (t > 1/2).$$

**Logistic map — stability of the period 2 orbit**

- A simple calculation gives

$$f_2'(x_\pm^*) = 4 + 2r - r^2.$$

- This is a decreasing function of $r$ for $r > 1$. $f_2'(x_\pm) = 1$ when $r = 3$ and $f_2'(x_\pm) = -1$ when $r = 1 + \sqrt{6} \approx 3.4494897$. $f_2'(x_\pm) < -1$ for $r > 1 + \sqrt{6}$.
- To see this, solve

$$4 + 2r - r^2 = -1, \qquad \text{equivalent to} \qquad r^2 - 2r - 5 = 0,$$

using the formula for a quadratic.

**Periodic orbits - general definitions**

- **Periodic orbit:** A periodic orbit of period $n$ is a sequence $\{x_0^*, \cdots, x_{n-1}^*\}$ for which

$$x_1^* = f(x_0^*), \;\; x_2^* = f(x_1^*), \;\; \cdots \;\; x_{n-1}^* = f(x_{n-2}^*), \;\; x_0^* = f(x_{n-1}^*),$$

with the property also that all the points are distinct from each other.
- By this definition a fixed point is also a period 1 orbit.
- **Stable orbit:** A periodic orbit is stable if each point on it belongs to some interval such that every orbit starting from an arbitrary point in the interval converges to the periodic orbit.

- Such an orbit will also be called a *periodic attractor*. Moreover, the set of initial values from which iterations converge to a periodic attractor are called its basin of attraction.
- **Composition of a map:** The $n$-fold composition $f_n$ of the map $f$ is defined inductively by

$$f_n(x) = f_{n-1}(f(x)).$$

- Obviously each member $x_k^*$ of a periodic orbit is a fixed point of the $n$-fold composition $f_n$. Because of the requirement that all the points be distinct, they cannot be fixed points of any composition $f_m$ for which $m < n$.

## Periodic table of orbits

- The above gives an alternative method of defining a periodic orbit; any fixed point $x^*$ of the $n$-fold composition $f_n$, which is not a fixed point of $f_m$ for all $m < n$, generates a period $n$ orbit.
- Here is a table of periodic orbits of the logistic map for $r = 3.7$. The second column gives the number of fixed points of $f_n$, obtained using *Chaos for Java*; the next is the number which belong to periodic orbits of period $m < n$ where $m$ divides $n$. The difference (which must be a multiple of $n$) is the number of fixed points which belong to period $n$ orbits.

| $n$ | fixed points | period $m < n$ | new points | period $n$ |
|---|---|---|---|---|
| 1 | 2 | – | 2 | 2 |
| 2 | 4 | 2 | 2 | 1 |
| 3 | 2 | 2 | – | – |
| 4 | 8 | 4 | 4 | 1 |
| 5 | 2 | 2 | – | – |
| 6 | 16 | 4 | 12 | 2 |
| 7 | 2 | 2 | – | – |
| 8 | 32 | 8 | 24 | 3 |

- Graphs for $1 \leq n \leq 7$ are shown in overheads.(Overheads 5_4 to 5_9)
- Check $n = 8$ for yourself using *Chaos for Java*.

## Stability of period $n$ orbits

- By the same argument used for period 2, stability of a period $n$ orbit depends on whether the absolute value of the derivative $f_n'$ is less than 1 (stable) or greater than 1 (unstable).
- We need a general formula for the derivative of an $n$-fold composition $f_n$. It's quite simple: if $x_i, x_{i+1}, \cdots, x_{i+n-1}$ are $n$ successive iterates of the map starting with $x_i$, then

$$f_n'(x_i) = \prod_{j=0}^{n-1} f'(x_{i+j}) = f'(x_i) \cdot f'(x_{i+1})) \cdots f'(x_{i+n-1}). \tag{5.2}$$

  This is the product of derivatives of $f$ at the $n$ successive iterates, starting from $x_i$.
- This ensures that the test — indeed the derivative value $f_n'$ — does not depend on particular points on the orbit. They all contribute once and once only.

## Derivative of $f_n$

- The demonstration of uses mathematical induction. I shall show that if it is true for some number $n$, then it is true for the next number $n+1$. It has already been proved for $n = 2$ in equation (5.1).
- For arbitrary $n$, $f_n(x) = f_{n-1}(f(x))$ and the same argument gives

$$\begin{aligned} f_n'(x_i) &= f_{n-1}'(f(x_i)) \cdot f'(x_i) \\ &= f_{n-1}'(x_{i+1}) \cdot f'(x_i) \\ &= f'(x_{i+n-1}) \cdots f'(x_{i+1}) \cdot f'(x_i), \end{aligned}$$

  as required.
- It follows that points on a periodic orbit all have the same stability, since the values which go into the calculation of $f_n'(x_i^*)$ come from each point of the periodic orbit.
- So we have a simple derivative test: a periodic orbit is stable, or unstable, according as the product of the derivatives $f'(x_i^*)$ taken over the orbit has magnitude less than, or greater than, unity.