Lecture 4 — Orbits of the tent map

Source material: Chapter 2, pp 24–29

To reproduce overheads shown in lectures, download the corresponding files from the website and open them with "Chaos for Java"

Switching behaviour — $t > 1/2$

- In the last lecture, I showed that, for any starting point, except the fixed points x_0^* and x_1^*, or the end point $x = 1$, iterations of the tent map will keep switching from one side of $x = 1/2$ to the other; they can never stay indefinitely on either side.
- I display 32 iterations of the tent map with $t = 0.9$, starting from $x_0 = 0.1$. (Overhead 4_1) The switching behaviour is clearly seen.
- The same argument helps explain the switching behaviour of the Lorenz attractor.

Period 2 orbit

- Consider the orbit of the tent map, with $t > 1/2$, and initial value

$$x_0^* = \frac{2t}{1 + (2t)^2}. \tag{4.1}$$

This satisfies $x_0^* < 1/2$ for any $t > 1/2$, since

$$\frac{1}{2} - x_0 = \frac{1 + 4t^2 - 4t}{2(1 + 4t^2)} = \frac{(1 - 2t)^2}{2(1 + 4t^2)} > 0.$$

- After one iteration,

$$x_1^* = 2tx_0^* = \frac{4t^2}{1 + 4t^2} > \frac{1}{2}. \tag{4.1b}$$

- For the next iteration,

$$x_2^* = 2t(1 - x_1^*) = x_0^*.$$

- So the system alternates between just two states,

$$x_0, x_1, x_2, x_3, \cdots = x_0^*, x_1^*, x_0^*, x_1^*, \cdots$$

This is known as a periodic, or period 2, orbit.

- Let's attempt to observe this orbit numerically for $t = 0.9$. To 9 decimal places, the formula for the orbit gives $x_0^* \approx 0.424528302$, $x_1^* \approx 0.764150943$.
- After about 30 iterations they separate, after 40 iterations they return to apparent chaos. There is no discernible relationship between the two orbits once this happens, apart from the continual switching from side to side. (Overheads 4_2 & 4_3)
- Lorenz describes the situation as follows (from his book, *"The Essence of Chaos"*)

  "Returning to chaos, we may describe it as behaviour that *is* deterministic, or is nearly so if it occurs in a tangible system that possesses a slight amount of randomness, but does not *look* deterministic. This means that the present state completely or almost completely determines the future, but does not appear to do so. How can deterministic behaviour look random? ... almost, but not quite, identical states occurring on two occasions will *appear* to be just alike, while the states that follow, which need not be even nearly alike, will be observably different. In fact, in some dynamical systems it is normal for two almost identical states to be followed, after a sufficient time lapse, by two states bearing no more resemblance than two states chosen at random from a long sequence. Systems in which this is the case are said to be *sensitively dependent* on initial conditions."
- The orbits shown differ not only in their initial states, but also in the effect of numerical error at each computational step.

2-fold composition map

- Iterating the map twice results in the sequence

$$x_0 \to x_1 = f(x_0) \to x_2 = f(x_1) = f(f(x_0)).$$

- I write a subscript to denote the function of a function $f(f(x))$ or *function composition* $(f \circ f)(x)$, i.e.,

$$f_2(x) = f(f(x)) = (f \circ f)(x),$$

  This is the *second composition* of the map f.
- The importance of this new map follows from the following observations:

(1) Fixed points of f must be fixed points of f_2: if $f(x^*) = x^*$, then

$$f_2(x^*) = f(f(x^*)) = f(x^*) = x^*.$$

(2) If x_1^* is a fixed point of f_2 which is not a fixed point of f, then $x_2^* = f(x_1^*)$ has the same property, because

$$f_2(x_2^*) = f_2(f(x_1^*)) = f(f_2(x_1^*)) = f(x_1^*) = x_2^*.$$

(3) Pairs of fixed points of f_2, x_1^*, $x_2^* = f(x_1^*) \neq x_1^*$ are period 2 orbits, since

$$f(x_2^*) = f_2(x_1^*) = x_1^*.$$

Visualising f_2

- Here is the graph for $t = 0.9$. (Overhead 4_4)
- We see the fixed point and the period 2 orbit quite clearly. Moreover, we can see that the derivative of f_2' satisfies $|f_2'(x^*)| > 1$ for each of them.
- There is no possibility for further periodic orbits of period 2, since there are only four fixed point of f_2. Two of these are the fixed points of f, the other two constitute the periodic orbit found above.

A simple map $\cdots$

- Setting $t = 1$, the dynamics is given by the formula

$$x_{k+1} = \begin{cases} 2x_k, & 0 \leq x_k \leq 1/2, \\ 2(1 - x_k), & 1/2 \leq x \leq 1. \end{cases}$$

  If we work with binary arithmetic, there is a very simple way to express this.

- Every number $0 \leq x \leq 1$ has a *binary representation*

$$x = 0.d_1 d_2 d_3 \ldots$$

  where the d_j are either 0 or 1. The meaning is that

$$0.d_1 d_2 d_3 \ldots = d_1/2 + d_1/2^2 + d_1/2^3 + \cdots$$

This representation is not unique; sequences ending with the recurring number 1 are equivalent to a terminating sequence, that is,

$$0.d_1d_2d_3\ldots d_k\,\dot{1} \equiv 0.d_1d_2d_3\ldots(d_k+1)\,0.$$

If $d_k = 1$, then $(d_k + 1) = 0$, carry 1; this process continues until the unit can be added without a carry. The extreme example is that $0.\dot{1} = 1.0$.

- Now it is easy to see that, for the map, a number with the base 2 representation $0.d_1d_2d_3\ldots$ is mapped to

$$f(0.d_1d_2d_3\ldots) = \begin{cases} 0.d_2d_3\ldots & (x \le 1/2) \\ 0.\bar{d}_2\bar{d}_3\ldots & (x \ge 1/2) \end{cases}$$

where $\bar{d}_j = 1 - d_j$ is the 2's complement. At first glance, it seems that I forgot about the first digit d_1; a little reflection shows that if $x \le 1/2$ we may take $d_1 = 0$, and if $x \ge 1/2$ we may take $d_1 = 1$ so that $\bar{d}_1 = 0$. So, the dynamics is a simple shift of one to the left, together with a complement operation if $d_1 = 1$.

$\cdots$ with complex behaviour

- From this analysis we can draw come surprising conclusions:

- *Sensitive dependence* — suppose we know the initial value x_0 to N binary places. Consider the (uncountable) collection of random numbers which agree with x_0 to this accuracy. Then, after N iterations, the orbit is essentially random, since it follows the binary representation of a random number in no way related to the initial condition.

- *Mixing* — consider the *interval* of initial values which differ first at the Nth binary place. At the Nth iteration, they are spread over the whole interval $[0, 1]$. Thereafter the whole interval is stretched to twice its length at each iteration and then folded back on itself.

- *Dense periodic orbits* — the binary representation of every *rational* number ends with a recurring group, and so generates an orbit which eventually repeats periodically. *Irrational* numbers have a binary representation which never recurs. Therefore, orbits with periodic behaviour are *dense* in the set of chaotic orbits.

- The conclusion is that the dynamics of even the most simple nonlinear system can be as pathological as the structure of the rational numbers in the reals.