Source material: Chapter 2, pp 19–23

To reproduce overheads shown in lectures, download the corresponding files from the website and open them with "Chaos for Java"

Lecture 3 — Stability of fixed points

- **Stable and unstable fixed point:** A fixed point x^* is stable if it belongs to an interval $I = (a, b)$, such that for any x_0 in I, the orbit which commences from x_0 converges to x^* as k increases toward infinity; it is unstable if it is ejected as k increases toward infinity.
- A stable fixed point will also be called an *attractor*, an unstable fixed point a *repeller*.
- A fixed point may also be stable in some weaker sense (the definition I have given is generally called asymptotic stability).
- **Basin of attraction:** The set of all initial states whose orbits converge to a given attractor is called the basin of attraction.
- A crucial part of these definitions is a knowledge of the behaviour of a dynamical system in the long run, expressed in mathematical terms by statements like "as k increases toward infinity".

Determination of stability

- How can one check the stability of a fixed point from the properties of the map alone? The answer is that it depends on the first derivative of the map.
- I shall assume that the maps we consider are *smooth* in the vicinity of any fixed point. By smooth, I mean that the function has continuous derivatives of all orders.
- Consider the stability of a fixed point x^* of a map f. Let $\delta_k = x_k - x^*$ denote the difference between the kth iterate and the fixed point. We want to investigate the distance, $|\delta_k|$, as k increases.
- Using linear approximation (Taylor expansion) and then taking absolute values,

$$\begin{aligned}|\delta_{k+1}| &= |f(x_k) - x^*| = |f(x^* + \delta_k) - x^*| \\ &\approx |f(x^*) + \delta_k f'(x^*) - x^*| = |f'(x^*)||\delta_k|.\end{aligned}$$

- How good is the approximation? The answer is, that for a smooth map, we may in fact write an *exact* equation

$$f(x^* + \delta_k) = x^* + \delta_k f'(x^{**}),$$

where x^{**} is a point on the smooth curve, no further away from x^* than δ_k.

- We have two possibilities of interest:

(1) $|f'(x)| < 1$ on the interval I. If we denote the *maximum* value of $|f'(x)|$ on I by M, then we have

$$|\delta_{k+1}| \leq M|\delta_k|.$$

- This says that the new iteration is within the interval I, and closer to x^* than the original, so we may iterate the argument, giving

$$|\delta_{k+l}| \leq M^l|\delta_k|,$$

after l steps. Since $M < 1$, the sequence converges to zero and the fixed point is stable.

(2) $|f'(x)| > 1$ on the interval I. Denote the *minimum* value of $|f'(x)|$ on I by M, then we have

$$|\delta_{k+1}| \geq M|\delta_k|.$$

- This says that the new iteration cannot be as close as the first! Moreover, no matter how close to x^* we start the iteration, this argument shows that it moves away by a magnification factor at least as big as M until it is thrown out of the interval.
- Now let's put all this together. Given a fixed point x^* of a smooth map, at which $|f'(x^*)| \neq 1$, then there is an interval containing x^* in which either $|f'(x)| < 1$ or $|f'(x)| > 1$, according as whether $|f'(x^*)| < 1$ or $|f'(x^*)| > 1$. So the stability of the fixed point is completely determined by the value of $|f'(x^*)|$.
- The last two overhead transparencies from lecture 2 show precisely the behaviour we have found theoretically.

Fixed points of the logistic map

- The fixed point equation $x = f(x)$ is the quadratic equation $x(rx - r + 1) = 0$ whose two solutions are $x_0^* = 0$ and $x_1^* = (r-1)/r$.
- For $0 < r < 1$, x_1^* is not in the interval $[0, 1]$ and there is only one fixed point of interest: x_0^*. Since $f'(0) = r$, this fixed point is stable.
- For $1 < r < 4$, there are two fixed points in $[0, 1]$, x_0^* (still) and x_1^* (new). Since $f'(0) > 1$ when $1 < r < 4$, x_0^* is unstable in the range $1 < r \le 4$. For the other fixed point x_1^*,

$$f'(x_1^*) = 2 - r.$$

So it is a stable fixed point in the range $1 < r < 3$, and unstable in the range $3 < r \le 4$.

Fixed points of the tent map

- For all $0 \le t \le 1$, one fixed point is $x_0^* = 0$. The derivative is $f'(0) = 2t$, so x_0^* is stable for $0 \le t < 1/2$ and unstable for $1/2 < t \le 1$. (Overhead 3_1)
- For $t > 1/2$, there is an additional solution given by the intersection of the lines $y = x$ and $y = 2t(1 - x)$: (Overhead 3_2)

$$x_1^* = \frac{2t}{1 + 2t}, \qquad t \ge 1/2. \tag{3.1}$$

The formula $f(x) = 2t(1 - x)$ only applies if $x \ge 1/2$, so we must insist that $x_1^* \ge 1/2$; this is equivalent to $t + 1/2 \le 2t$, or $t \ge 1/2$. As for stability, $f'(x_1^*) = -2t$.

- Therefore x_0^* and x_1^* are both unstable for $1/2 < t \le 1$.

Role of the fixed points

- For $0 \le t < 1/2$, any point x_0 in the interval $[0, 1]$ is mapped on the first iteration to a point x_1 in the interval $[0, 1/2)$, since the maximum of the map is $f(1/2) = t$. Thereafter the dynamics follows the simple rule

$$x_k = (2t)^{k-1} x_1, \tag{3.2}$$

so all orbits converge to the fixed point $x_0^* = 0$. The map is boringly stable!

- For $t > 1/2$, the tent map is totally unstable. In this case any initial point x_0 in the interval $(0, 1/2)$ is repelled by the fixed point x_0^* according to (3.2), until it attains a value $x > 1/2$, where the map is defined by the different formula $f(x) = 2t(1 - x)$.
- It is instructive to rewrite this in a way which shows the role of the fixed point x_1^*. Using equation (3.1), and a little algebra, gives

$$(x_{k+1} - x_1^*) = (-2t)(x_k - x_1^*).$$

Therefore the iterations alternate from one side of x_1^* to the other while getting further away, each time by the factor $2t > 1$, until iterations re-enter the interval $(0, 1/2)$.

Fixed points of the Cubic #1 map

- The fixed point equation is

$$27rx^{*2}(1 - x^*)/16 = x^*,$$

which has three solutions (it's a cubic).
- One solution is zero, I denote it $x_0^* = 0$. Note that

$$f'(x_0^*; r) = 0, \qquad \text{for all values of } r,$$

so this is a stable fixed point for all values of r.
- The other two solutions come from cancelling the common factor x^* in the cubic, to give the quadratic equation

$$x^{*2} - x^* + \frac{16}{27r} = 0.$$

- Using the formula for roots of a quadratic, the other two solutions, which I denote $x_\pm^*$, are

$$x_\pm^* = \frac{1 \pm \sqrt{1 - 64/27r}}{2} \tag{3.3}$$

These solutions are not real if $64/27r > 1$, but they are real (and therefore new fixed points) when $64/27r \leq 1$ which is equivalent to

$$r > r^* = 64/27 \approx 2.37.$$

Stability

- A formula may be given for $f'(x^*_\pm; r)$ by substituting the formula for $x^*_\pm$ into the formula for f', this gives a complicated function of r. It's not the way to go!
- Let's approach the problem of investigating stability of the fixed points $x^*_\pm$ indirectly by asking the question: given a number d, can we find fixed point(s) x^* for which $f'(x^*; r) = d$? If so, we can ask what values of r lead to d in the range

$$-1 < d < 1.$$

- We require the solution of two simultaneous equations:

$$27r(x^* - x^{*2})/16 = 1,$$

$$27r(2x^* - 3x^{*2})/16 = d.$$

The first is for the fixed point (note that I cancelled the factor x^* which gave x^*_0 since we know that $d = 0$ for it); the second is the equation for the derivative.

- Multiplying the first equation by d, subtracting the second, then dropping the multiplying factor $27r/16$ from what results (since the RHS is now zero, while $27r/16$ is a common factor on the LHS), gives the much simpler equation

$$(d - 2)x^* - (d - 3)x^{*2} = 0,$$

and the non-zero solution is

$$x^* = (d - 2)/(d - 3).$$

- For stability, we are interested in values of d between -1 and 1, which gives values of x^* between 3/4 (for $d = -1$) and 1/2 (for $d = 1$). Substituting these values we find that

$$\left.\begin{matrix} d = 1 \\ d = -1 \end{matrix}\right\} \quad \text{when} \quad \left\{\begin{matrix} x^* = 1/2 \\ x^* = 3/4 \end{matrix}\right\} \quad \text{when} \quad \left\{\begin{matrix} r^* = 64/27 \\ r^* = 256/81 \end{matrix}\right.$$

Also, $1 < d < 2$ if $0 < x^* < 1/2$ and $d < -1$ if $r > 256/81$.

- So we have shown

(1) x^*_+ is stable for $64/27 < r < 256/81$, unstable for $r > 256/81$.

(2) x^*_- is unstable for all $r > 64/27$.