Lecture 22 — A case study

To reproduce overheads shown in lectures, download the corresponding files from the website and open them with "Chaos for Java"

Potential energy

- The driven non-linear oscillator equation (20.3) is quite general. What is represents depends on the restoring force $-f(\theta)$.
- In engineering and the physical sciences, it is usual to think rather of a *potential energy* function, whose derivative (gradient) gives the restoring force:

$$f(\theta) = \frac{dU(\theta)}{d\theta}.$$

- For the linear case (harmonic oscillator),

$$f(\theta) = -k\theta, \qquad U(\theta) = \frac{k}{2}\theta^2.$$

Because the restoring force grows without bound, motion is always bounded. The potential energy function, being a parabola, is a *potential well.*

Escape from a potential well

- This is not possible in the simple linear case.
- If the potential energy has only a local minimum at $\theta = 0$, and is bounded by a hill, then escape is possible.
- This is the case for the driven pendulum, for which

$$f(\theta) = k\sin(\theta), \qquad U(\theta) = -k\cos(\theta).$$

The function $-\cos(\theta)$ has a local minimum at every $\theta = 2n\pi$, and a local minimum at every $\theta = (2n+1)\pi$; escape from one minimum to the next is just rotation.

- In many interesting systems, escape is a one-off event — the system transits to a totally different state. In the case of a ship, it might be better called a catastrophe, since it presumably represents capsize!

- Two simple models are obtained by adding either a cubic or a quartic term to the harmonic oscillator:

$$\frac{1}{2}\theta^2 - \frac{1}{3}\theta^3 \qquad \text{or} \qquad \frac{1}{2}\theta^2 - \frac{1}{4}\theta^4.$$

- The first is a symmetric function, and represents the situation where escape to either side is equally likely.
- The second is asymmetrical, with escape occurring only to positive values of θ.
- The multiplying factors have been chosen so that the top of the potential hill is at $|\theta| = 1$. This defines the unit of measurement for θ.

A canonical escape equation

- In this lecture, I want to discuss the asymmetrical case. As a physical example, one might imaging the rolling of a ship, where the asymmetry is due to a side wind, which is insufficient to capsize the ship by itself.
- Setting this into equation (20.3), and using a simple trigonometric driving term as in (20.1), we have our simple model:

$$\theta'' + \gamma\theta' + \theta - \theta^2 = k \cos \Omega t,$$

The driving force is supposed to model the effect of regular wave action.

- Remember that time is measured relative to the frequency of small amplitude free oscillations, θ in units just defined, so the factors γ, k, and Ω will be relative to this.
- I follow work done of Thompson and others, at University College, London, in setting

$$\gamma = 0.1,$$

a reasonable value in studies of ships made by them.

- Clearly, the effect of the waves is likely to be greatest for values of $\Omega \approx 1$, I again follow Thompson in setting

$$\Omega = 0.85,$$

as an interesting case for investigation.

Sympathetic motion

- For small wave loading (k value) there should be a periodic attractor at the driving frequency Ω, observable as a stable fixed point of the Poincaré section.
- There should also be a basin of attraction, and a basin of infinity representing capsize.
- The boundary between the two should be a hyperbolic fixed point of the Poincaré section, related to the top of the potential hill.
- So the ship simply rocks in sympathy with the waves (Overheads 22_1 & 20_2).
- More importantly, so long as the ship starts in the basin of attraction, it settles to this motion. This means that, in the real-life situation of continual change, the ship is able to adapt to them by natural forces alone.
- It is not just stable in the narrow sense of linear stability of the fixed point, it is dynamically stable so long as it stays in a safe basin of attraction.

The soft spring

- The amplitude of oscillation, typified by the position of the corresponding fixed point, increases as the wave amplitude k increases.
- This increase is not linear. In fact, as the amplitude increases, the well is less effective as a restoring mechanism. The can be measured by the fact the width of the well gets larger than that of the linear oscillator, for which the amplitude is proportional to k.
- For example, let's calculate the distance from the top of the hill to the other side. The top of the hill is at $\theta = 1$, where $U(1) = 1/2 - 1/3 = 1/6$. The other place where U takes this value is $\theta = -1/2$, giving a width of $3/2$ for a potential barrier of height $1/6$.
- In comparison, the harmonic oscillator, $U(\theta) = \theta^2/2$, takes the value $1/6$ at $\theta = \pm 1/\sqrt{3}$, and we have

$$\text{barrien width at } U = 1/6 \qquad 1.5 \text{ (cubic)}, \qquad 1.155 \text{ (quadratic)}.$$

- The restoring force is *soft*, so the amplitude should increase faster than linearly. This is seen in the following table, at least for values of k up to ~ 0.68.

Period 1 orbits — Anharmonic oscillator

k	x_0	x_1	x_2	x_3
0.000	-1.000U	0.000S		
0.010	-1.006U	0.033S		
0.020	-1.011U	0.065S		
0.030	-1.017U	0.098S		
0.035	-1.020U	0.115S		
0.040	-1.023U	0.132S		
0.045	-1.026U	0.150S		
0.047	-1.027U	0.157S	–	–
0.0472	-1.027U	0.158S	-0.134U	-0.174S
0.0475	-1.027U	0.159S	-0.081U	-0.226S
0.048	-1.027U	0.161S	-0.037U	-0.268S
0.049	-1.028U	0.165S	0.017U	-0.319S
0.050	-1.028U	0.169S	0.056U	-0.356S
0.055	-1.031U	0.188S	0.172U	-0.466S
0.060	-1.034U	0.210S	0.232U	-0.529S
0.065	-1.037U	0.235S	0.265U	-0.574S
0.067	-1.038U	0.247S	0.271U	-0.588S
0.068	-1.038U	0.255S	0.272U	-0.595S
0.069	-1.039U	–	–	-0.602S
0.070	-1.040U			-0.608S
0.075	-1.042U			-0.636S
0.080	-1.045U			-0.659S
0.090	-1.051U			-0.696S
0.100	-1.056U			-0.725S
0.110	-1.061U			-0.750U

Bifurcations and hysteresis

- Measured in terms of the ratio of response to disturbance, the non-linearity does not seem to be a major issue. The ratio x^*/k increases from 0.33 to 0.36 as k increases to $k \approx 0.065$ — not a dramatic change!
- What is dramatic is the *qualitative* dependence of the period-1 orbits on k, viewed as a bifurcation diagram of the map.
- Because of the complexity and length of the computations, *Chaos for Java* can not calculate these diagrams. We must do it ourselves, by gathering data as in the table.
- In fact, the fixed point, which is initially stable, traces out a *smooth curve* in x-k space! But it makes an "S-bend", changing stability at each "change of direction".

- A new pair of orbits are born at $k \approx 0.472$ in a tangent bifurcation, one unstable, the other another stable attractor. (Overheads 22_3 & 22_4)
- The original stable attractor dissapears, at $k \approx 0.068$, when it joins the unstable orbit, in a *reverse tangent* bifurcation.
- It is the large-amplitude attractor which survives to larger k-values, until it period-doubles somewhere between $k = 0.1$ and $k = 0.11$.
- At the tangent bifurcations, one eigenvalues of the stability matrix attains the value $+1$; at the period doubling, one passes through the value -1.
- If k is gradually increased from small values, then the system tracks the small-amplitude response until critical value $k \approx 0.068$ is reached, when jumps to the large-amplitude response.
- If now k is gradually decreased, the system tracks the large-amplitude response until the critical value $k \approx 0.047$ is reached, and again there is a jump.
- This is an example of *hysteresis*.

Basins of attraction - evolution

- Overheads 22_5 & 22_6 show the fixed points, and the basins of attraction, for $k = 0.05$ and $k = 0.06$.
- It is seen that the basin of the small-amplitude response shrinks, while the basin of the other grows. The boundary between them is the stable manifold of the hyperbolic fixed point.
- What this means in a dynamically changing situation, is that the ship can quite suddenly jump from small-amplitude to larger-amplitude rolling, and that, as the wave loading increases, it is more likely to lock on to the latter.

Basins of attraction - fractal invasion

- There is worse to come: Quite quickly, the basin boundary becomes fractal, and indeed fractally invaded. (Overheads 22_7 & 22_8)
- Basically, the possibility of safe operation just disappears ! It only takes much smaller

disturbances to jolt the system from one part of the basin of attraction to a part of the basin of infinity which is now occupying much of the former safe operating region.

- After that, a few lurches, which don't seem particularly different from the stable attractor, and then — over she goes!
- What makes the basin boundary fractal? The technical answer is a homoclinic tangle. The outgoing (unstable) and ingoing (stable) manifolds of the fixed point on the boundary get to intersect.
- But that means they intersect infinitely often, which can only be if they are infinitely tangled.
- An easy visual picture is that the basin of attraction gets stretched and folded as the phase of section varies continuously.
- I show sections taken at 15 degree intervals. They may be viewed, as an animated gif, on my website, under the heading "Poincaré Sections".
- My last overhead, also not downloadable, is taken from an article by J. M. T. Thompson. It shows a similar set of calculations made for a model of the Danish Tanker "Edith Terkol", a ship which is sadly no longer with us following a capsize. The pictures are symmetrical because the study was done in the absense of wind loading.