Lecture 21 — Poincaré sections

Source material: Chapter 6, pp 186–196

To reproduce overheads shown in lectures, download the corresponding files from the website and open them with "Chaos for Java"

Poincaré sections

- Even the phase plane picture is complicated, in fact you can examine chaotic orbits in the phase plane without gaining much insight, except for the obvious complexity.
- For differential equations of the type under consideration, it is easy to implement the method of *Poincaré sections.*
- In general, the difficulty is to find a suitable surface which the orbit must pierce repeatedly; for a periodically driven system the phase plane provides such a surface.
- It is pierced once, and once only, for each value of the time, allowing us to record the positions at a convenient sequence of times.
- There is an obvious choice for these times, namely once per cycle of the applied force, always at the same point of the cycle:

$$t_n = \frac{\phi_0 + 2\pi n}{\Omega}, \qquad n = 0, 1, \cdots.$$

- The constant ϕ_0 is the *phase* of the section, and may be varied to get different sections.

The period map

- What is so special about the choice of t_n?
- It comes from an important property of the differential equations (20.4), namely

$$\text{if } \big(\theta(t), \omega(t)\big) \text{ is a solution, then so is } \big(\theta(t + 2\pi/\Omega), \omega(t + 2\pi/\Omega)\big).$$

- In general, the two solutions will not be the same.
- Let's call them $(\theta_0(t), \omega_0(t))$ and $(\theta_1(t), \omega_1(t))$; they are related by the simple formulae

$$\theta_1(t) = \theta_0(t + 2\pi/\Omega), \qquad \omega_1(t) = \omega_0(t + 2\pi/\Omega).$$

- The first pair, $(\theta_0(t), \omega_0(t))$, is the unique solution with initial values

$$\theta_0(t_0) = \theta_0, \qquad \omega_0(t_0) = \omega_0,$$

and after one time step they have the values

$$\theta_1 = \theta_0(t_0 + 2\pi/\Omega), \qquad \omega_1 = \omega_0(t_0 + 2\pi/\Omega).$$

- These are the initial values for the solution $(\theta_1(t), \omega_1(t))$, starting from $t = t_0$; the two solutions are in fact part of the same (infinite) orbit, one period out of step.

Phase of section

- An initial time t_0 is specified by an initial phase $\phi_0 = \Omega t_0$, which may be incremented by any integer multiple of 2π. This leads to:
- **Phase of section:** Given an initial time t_0, the corresponding phase of section is given by

$$\phi_0 = \Omega t_0, \qquad \bmod\ 2\pi,$$

meaning that ϕ_0 is reduced by an integer multiple of 2π so that $0 \le \phi_0 < 2\pi$.

- For any initial point (θ_0, ω_0), starting at initial phase ϕ_0, the differential equations determine a unique orbit, which arrives at a point

$$\big(\theta_1, \omega_1\big) = \big(\theta_0(t_0 + 2\pi/\Omega), \omega_0(t_0 + 2\pi/\Omega)\big) = \big(\theta_1(t_0), \omega_1(t_0)\big),$$

after one period of the driving force.

- This is a two-dimensional map of the phase plane to itself, in which the phase appears as a parameter.
- So we may reduce the investigation from three to two dimensions.
- **Poincaré sections:** Given a phase of section ϕ_0, the orbit of the map

$$(\theta_0, \omega_0) \to (\theta_1, \omega_1) \to \cdots \to (\theta_k, \omega_k) \to \cdots$$

generated by sampling positions in the phase plane at the sequence of times

$$t = t_0 \to t_1 = t_0 + 2\pi/\Omega \to \cdots \to t_k = t_{k-1} + 2\pi/\Omega \to \cdots$$

is a Poincaré section of the corresponding orbit of the differential equation.

- Poincaré sections for two orbits from lecture 20 are shown in Overheads 21_1 & 21_2.
- It is immediately clear that the first orbit is period 2, meaning that its period is twice that of the driving force, and that there is a strange attractor in the chaotic case.

Periodicity

- Any orbit which is periodic, with the same period as the driving force, will be a fixed point of the Poincaré section.
- Conversely, if an orbit is a fixed point of the Poincaré section, then it is periodic as a function of time, satisfying

$$\theta(t + 2\pi/\Omega) = \theta(t), \qquad \omega(t + 2\pi/\Omega) = \omega(t),$$

- More generally, if a solution determines a period n orbit of the Poincaré section (as a map), then it is periodic as a function of time, satisfying

$$\theta(t + 2\pi n/\Omega) = \theta(t), \qquad \omega(t + 2\pi n/\Omega) = \omega(t),$$

  and conversely.
- This means that Poincaré sections contain complete information about the periodicity or otherwise of the corresponding full solution, although the frequency information is restricted to frequencies no greater than $\Omega/2$.
- So the question of long term periodicity can be settled from the Poincaré section alone.
- In particular, if the discrete Fourier spectrum of the Poincaré section contains non-zero amplitudes at all frequencies in the range $0 \le \omega \le 1/2$, even after discarding a sufficiently long initial transient, then the orbit cannot be periodic.
- Overhead displays Fourier spectra corresponding to the Poincaré sections of the Overheads 21_1 & 21_2 are shown in Overheads 21_3 & 21_4.

Lyapunov exponents

- Lyapunov exponents may be defined for the solutions of differential equations and for Poincaré sections, in the latter case there are two independent Lyapunov exponents, in the former case three. Numerical calculation of either is rather complicated.
- It is natural that Lyapunov exponents for differential equations should be expressed as averages per unit time. This leads to the following definition:
- **Lyapunov exponents of Poincaré section:** Given an initial point (x_0, y_0), the Lyapunov exponents $L_1(x_0, y_0)$, $L_2(x_0, y_0)$ of a Poincaré section are given by the formulae

$$L_1(x_0, y_0) = \lim_{k\to\infty} \frac{1}{T_k} \left(\lim_{\delta_0\to 0} \ln |\delta_k^{\max}/\delta_0| \right),$$
$$L_2(x_0, y_0) = \lim_{k\to\infty} \frac{1}{T_k} \left(\lim_{\delta_0\to 0} \ln |\delta_k^{\min}/\delta_0| \right),$$

  where T_k is the elapsed time for the k iterations.
- For the systems under consideration, the time is given by

$$T_k = \frac{2\pi k}{\Omega},$$

  this is a rescaling by the factor $\Omega/2\pi$.
- There is an important feature in common between Poincaré sections of the driven nonlinear system (20.4) and the generalised Hénon map; both are dissipative maps, with constant area contracting factor.
- For the Hénon map, this factor is the absolute value $|b|$, in the present case it is given by

$$\exp(-2\pi\gamma/\Omega). \tag{21.1}$$

- An important consequence is that the two Lyapunov exponents are simply related. Taking into account the time rescaling, equation (21.1) gives the relation

$$L_1 + L_2 = -\gamma.$$

Lyapunov dimension

- For the Poincaré section, if $L_1 > 0$, dissipation ensures that $L_1 + L_2 < 0$, and the Lyapunov dimension is

$$d_L = 1 - L_1/L_2.$$

- In conclusion, note that there are two possibilities for the Lyapunov dimension of an attractor:

(i) A periodic attractor, which is a one-dimensional curve in three-dimensional space, or a zero-dimensional set of points in the phase plane. This has both (non-zero) Lyapunov exponents negative.

(ii) A strange attractor, which has $L_1 > 0$, $L_2 < 0$, and Lyapunov dimension $d_L = 1 - L_1/L_2 > 1$. Since L_2 is finite, this is a strict inequality.

- The missing possibility is $d = 1$, that is, an attracting solution which traces out a smooth surface in three-dimensions; this would appear in the Poincaré section as a smooth curve.

Strange attractor of the driven pendulum

- Some typical pictures are shown in Overheads 21_5 to 21_12. The stretching and folding, as the phase ϕ_0 is increased, is clearly seen, as is the symmetry of the attractor.
- Estimated values of the Lyapunov exponents, calculated on the same orbit data

$$L_1 \approx 0.117, \qquad L_2 \approx -0.617.$$

- Note that the sum of the two is $-\gamma = -0.5$ (to the quoted accuracy); any failure would be of the numerics rather than the theory.
- From these values

$$d_L \approx 1 + 0.116/0.616 \approx 1.19.$$

  Taken together with the Fourier spectrum, this data leaves little doubt that the observed behaviour is chaotic, the attractor strange.

- The Lyapunov exponents for the differential equations (there are three of them) satisfy a constraint that one of them is precisely zero. The other two coincide with the exponents obtained from Poincaré sections.