Lecture 20 — Non-linear Oscillations

Source material: Chapter 6, pp 177–185

To reproduce overheads shown in lectures, download the corresponding files from the website and open them with "Chaos for Java"

Differential equations

- Differential equations are an important class of dynamical systems.
- In these concluding lectures I consider second order equations with linear damping but non-linear restoring term, driven by a periodic applied force.

The driven non-linear pendulum

- As a dynamical model it represents a system consisting of a mass m constrained to move in a vertical plane at a constant distance l from a fixed pivot point.
- Because the mass moves on a fixed vertical circle, only one coordinate is needed to specify its state, the angular position $\theta(t)$, measured from the bottom (rest) position.
- Three forces combine to produce the motion:

(i) Gravity, acting vertically downward with magnitude mg; the tangential component $mg \sin\theta$ acts as a restoring force.

(ii) A damping force $c\theta'$, proportional to the angular velocity θ'.

(iii) An applied periodic driving force, $g(t)$, acting tangentially.

- The equation of motion is given by *Newton's second law* (mass × acceleration = force), which leads to the second order non-linear differential equation

$$ml\theta'' + c\theta' + mg\sin\theta = g(t).$$

Dimensionless parameters

- One constant just sets the overall scale, since nothing changes if we divide through by (for example) the mass m.
- If we change the unit of time by the substitution $t \to t\tau$, (τ a constant) then the equation

becomes

$$ml\tau^{-2}\theta'' + c\tau^{-1}\theta' + mg\sin\theta = g(t).$$

- The natural choice for the measurement of time is made by setting $ml\tau^{-2} = mg$, giving

$$\tau = \sqrt{l/g}.$$

- $2\pi\tau$ is the period of small amplitude oscillations in the case of no damping and no external force, $1/\tau$ the corresponding angular frequency, $1/2\pi\tau$ the usual frequency.
- Making this choice, and dividing through also by mg,

$$\theta'' + \gamma\theta' + \sin\theta = g(t),$$

which leaves only two important quantities to be chosen.

- For simplicity, take $g(t)$ to be a trigonometric function, of angular frequency Ω and amplitude k; i.e.,

$$\theta'' + \gamma\theta' + \sin\theta = k\cos\Omega t. \tag{20.1}$$

Linear approximation

- The *small amplitude approximation* is obtained by making the linear (small θ) approximation $\sin\theta \approx \theta$, to give the constant coefficient equation,

$$\theta'' + \gamma\theta' + \theta = k\cos\Omega t.$$

- We can find a *particular solution* $\theta_p(t)$ of this equation by simple guesswork, sometimes called variation of parameters:

$$\theta_p(t) = A\cos(\Omega t + \delta). \tag{20.2}$$

- A is the *amplitude*, proportional to k, and δ is the *phase shift*, independent of k. Both depend on the damping factor γ and the frequency Ω.
- It is not difficult to check that this is indeed a solution of the equation, and to obtain formulae for A and δ.

A periodic attractor — linear system

- The solution $\theta_p(t)$ is not just periodic, if $\gamma > 0$ it is a periodic attractor.
- This is easy to demonstrate; given the particular solution (20.2), substitute the formula

$$\theta(t) = \theta_p(t) + \theta_h(t)$$

  into the differential equation.
- The terms in θ_p result in a cancellation of the driving term $k \cos \Omega t$, leaving the *homogeneous equation*

$$\theta_h'' + \gamma\theta_h' + \theta_h = 0.$$

- This is a constant coefficient equation, with solutions

$$\theta_h(t) = \exp(-\gamma t/2)\big(A\cos(\omega_0 t) + B\sin(\omega_0 t)\big),$$

  which tend to zero as $t \to \infty$, leaving θ_p as the unique attractor for the linear system.

A periodic attractor — non-linear system

- We expect that the linear approximation is not too bad for small amplitude.
- That is, we expect the full non-linear system should have a periodic attractor for values of k that are not too large.
- Overheads 20_1 & 20_2 show the solution which starts from $\theta(0) = \theta'(0) = 0$, driven at frequency $\Omega = 2/3$, with $\gamma = 1/2$, $k = 1/2$.

The butterfly effect

- Overhead 20_3 shows two solutions with $k = 1.5$, starting from $\theta(0) = 0$. The difference between them is that one has initial condition $\theta'(0) = 1.000$, the other $\theta'(0) = 1.001$.
- It was precisely this kind of qualitative difference between two solutions which differed by only a small change of initial condition, which gave birth to Lorenz' celebrated results.
- The paradox is that, even though the two solutions become so unrelated globally, there is at every place patterns of behaviour which can almost be matched.
- This is an indication that the solutions may be under the control of a strange attractor.

More general driven system

- Before proceeding, I want to generalise equation (20.1) slightly to include an arbitrary restoring term, that is, to consider the second order differential equation

$$\theta'' + \gamma\theta' + f(\theta) = g(t), \tag{20.3}$$

  where f is a smooth function of θ.

- A useful example is the *driven Duffing equation*,

$$f(\theta) = \alpha x + \beta x^3.$$

  The behaviour of this system depends critically on whether α is positive, negative, or zero, and also on the sign on β. Some important special cases are

(i) *Ueda oscillator* ($\alpha = 0$ — rescale θ to make $\beta = 1$)

$$\theta'' + \gamma\theta' + \theta^3 = g(t),$$

(ii) *Two-well oscillator* ($\beta < 0$ — rescale t and θ to make $\alpha = 1$, $\beta = -1$)

$$\theta'' + \gamma\theta' + \theta(1 - \theta^2) = g(t),$$

- Another important case, which I shall discuss in the final lecture in connection with ship stability and capsize, is the *anharmonic oscillator*,

$$\theta'' + \gamma\theta' + \theta - \theta^2 = g(t),$$

Phase Plane

- It is extremely useful to put the position and velocity on an equal footing.
- A good reason may be found in the fact that the initial values of both θ and θ' must be specified in order to uniquely determine the solution.
- To avoid giving one of the two a privileged rôle, introduce a separate symbol ω for the velocity, leading to

$$\begin{aligned} \theta' &= \omega, \\ \omega' &= -\gamma\omega - f(\theta) + g(t). \end{aligned} \tag{20.4}$$

- This pair of first order differential equations is the preferred form — its solutions live in a two-dimensional *phase plane* whose coordinates are θ, ω, or simply θ, θ'.

Properties of orbits

- Given an arbitrary initial time t_0, and arbitrary initial values θ_0, ω_0, the equations have a unique solution which satisfies

$$\theta(t_0) = \theta_0, \qquad \theta'(t_0) = \omega(t_0) = \omega_0.$$

- These orbits are smooth curves in three-dimensional space-time.
- Uniqueness means that they may never intersect, since a point of intersection would provide a set of initial conditions from which two different orbits emanate.
- Viewed in the phase plane, however, the curves may intersect; such points of intersection are places where $\theta(t)$ and $\omega(t)$ have identical values, but at different values of t.
- I return now to the driven pendulum.
- Overheads 20_2 shows an orbit in which the three-dimensional form is apparent.
- What is not quite so apparent is the rapid convergence to a periodic attractor, seen more clearly in a phase plane view such as those of Overheads 20_4 & 20_5, particularly the latter which shows the solution in the time interval $300 \le t \le 350$.
- It is important to realise that all the views of this solution were produced by looking at the same data from different aspects.

Period doubling

- For sufficiently large k, the non-linearity causes qualitative changes in the behaviour of the solutions.
- In order to concentrate on the attractor, one looks for evidence that the solution has converged and discards as transient the solution thus far.
- I have done that in Overheads 20_6 & 20_7, for $k = 1.0$ and $k = 1.07$, with the same values of γ and Ω as before.
- The results show clear evidence of period doubling, demonstrating the ubiquitous nature of that bifurcation.