Source material: Chapter 2, pp 13–19

To reproduce overheads shown in lectures, download the corresponding files from the website and open them with "Chaos for Java"

Lecture 2 — Unimodal maps

- **One-dimensional system:** An equation of the form

$$x_{k+1} = f(x_k; \mu), \tag{2.1}$$

  is called a discrete one-dimensional dynamical system.

- The quantity x is called the state variable; the coefficient μ, which is not affected by the iteration, is called a control parameter.
- A one-dimensional system has only a single state variable; some have more than one parameter.
- The function f must have the property that the domain (input) space is *mapped* to (into or onto) itself, so as to allow for iteration. I refer to them as *maps* rather than functions.
- **Orbit:** The sequence of values x_k, $k = 0, 1, \cdots$, generated by the system (2.1) is called an orbit of the system, while the value x_0 from which the orbit commences is called the initial state.
- x_0 completely determines the orbit, at least *in theory.*

Two important systems

- I shall show that different models often have identical patterns of behaviour which come from simple properties of the function. For this reason, I shall treat two special one-dimensional examples in much detail, knowing that there are general lessons to be learned. The two are:

(1) The logistic map, already met,

$$f(x) = rx(1-x), \qquad (0 \le r \le 4). \tag{2.2}$$

The state variable is x, the control parameter is r.

(2) The tent map,

$$f(x) = \begin{cases} 2tx, & (x \le 1/2), \\ 2t(1-x), & (x \ge 1/2), \end{cases} \qquad (0 \le t \le 1).$$

Again the state variable is x, but the control parameter is t.

- Graphs of the two, at their maximum parameter values, are shown (Overheads 2_1 & 2_2). Provided the parameters are restricted as indicated, each is a map of the interval $0 \le x \le 1$ to itself.

Maximum in z sequence — Lorenz equations

- We have already examined some orbits of the Lorenz equations. On each circuit they attain a maximum value for the variable z.
- It follows from the last of the equations (1.1) that these maxima must occur when

$$\frac{dz}{dt} = xy - bz = 0.$$

That is, maxima in z mark the places at which the orbit pierces the surface $z = xy/b$.

- The intersection with this transverse surface is a "thick line", and the information about the points is almost contained in the maximum z values alone, z_k.
- For a one-dimensional discrete system $z_{k+1} = f(z_k)$, if one plots pairs (z_k, z_{k+1}) in a plane, they will fall on the graph of f. Therefore, it is instructive to examine such a plot for the Lorenz equations.
- This was done by Lorenz in his original paper. (Overhead 2_3)

Unimodal maps

- On $0 \le x \le 1$, the function (2.2) has zero minima at the ends and a maximum of $1/4$ at the mid-point — similarly for the tent map.
- Therefore these maps fold the interval; they are increasing functions of x up to some point $x = x_{\max}$, after which they are decreasing.
- When $r > 2$ (for the logistic map) or $t > 1/2$ (for the tent map) they have a stretching action as well: the total length of the image is greater than the length of the original interval, that is, $2f(x_{\max}) - f(0) - f(1) > 1$.

- **Unimodal map:** A unimodal map of the interval $I = [a, b]$ is one which has a single maximum $x_{\max}$ in I.
- In elementary calculus, a critical point is defined as one where the derivative either has the value zero, or is not defined.

Examples of iteration

- Let's look at what happens when we iterate the logistic map, starting from the initial value $x_0 = 0.1$, with the parameter choice $r = 2.95$. (Overheads 2_4 & 2_5).
- One is a plot of x_k as a function of k, in which the actual values are joined by straight lines simply as a guide to the eye.
- The other is a *cobweb plot*, in which each vertical line guides the eye from x_k to $f(x_k)$, each horizontal line from $f(x_k)$ to x_{k+1}.
- The cobweb plot shows most clearly what is going on: the state of the system approaches the limiting point at which the graph of $y = f(x)$ intersects the line $y = x$.

The Cubic #1 map

- Let's look at a somewhat different unimodal map, (the "cubic #1 map in *Chaos for Java*)

$$f(x) = 27rx^2(1-x)/16, \qquad (0 \le r \le 4) \tag{2.3}$$

- Let's show that it is a unimodal map of the interval $[0, 1]$ for the parameter range $0 \le r \le 4$, with the chosen normalisation factor.
- Write

$$f(x) = crx^2(1-x),$$

where c is some constant to be fixed, then

$$f'(x) = cr(2x - 3x^2).$$

- Since everything is smooth on $[0, 1]$, the maximum (or minimum?) is at the critical point x_{crit} where $f' = 0$, that is

$$cr(2x_{crit} - 3x_{crit}^2) = 0; \qquad \text{implies} \qquad x_{crit} = 2/3.$$

- Since $f(0) = f(1) = 0$ and

$$f(x_{crit}) = f(2/3) = 4cr/27,$$

  x_{crit} is a maximum when cr is positive.

- We want f to be a unimodal map of the interval $[0, 1]$ for the parameter range $0 \leq r \leq 4$, it's maximum value is largest when $r = 4$ with c positive, being

$$f(x_{crit}; 4) = 16c/27 \qquad \text{which requires} \qquad c = 27/16 \qquad \text{for} \qquad f(x_{crit}; 4) = 1.$$

- This shows the reason for the factor 27/16 — it ensures that this is a map of the interval $[0, 1]$ for the same range of parameter values r as for the logistic map $(0 \leq r \leq 4)$.
- You can easily check that the maximum value is 1 when $r = 4$.
- Because of the x^2 term, there may be two non-zero intersections of the graph with the line $y = x$: for $r = 2.9$ they are at $x \approx 0.2863$ and $x \approx 0.7137$.

Some iterations

- It's interesting to look at cobweb iterations using *Chaos for Java*.
- In the first, I started the iteration from the initial value $x_0 = 0.3 > 0.2863$, which repels the iterations (Overhead 2_6).
- After a few iterations it is attracted to the other point of intersection.
- In the second, I started the iteration from the initial value $x_0 = 0.25 < 0.2863$ — it is attracted by $x = 0$ (Overhead 2_7).
- For the logistic map, $x = 0$ acted as a repeller and was hardly noticed; for this new map it is a point of attraction so it makes its presence felt.

Fixed points

- Points of intersection in the x-y plane of the two curves $y = f(x)$ and $y = x$ are evidently of great importance, which warrants some definitions.
- **Fixed point:** Any value x^* for which $f(x^*) = x^*$ is called a fixed point of f.