Lecture 19 — Hénon and Rössler attractor

Source material: Chapter 5, pp 167–173

To reproduce overheads shown in lectures, download the corresponding files from the website and open them with "Chaos for Java"

Hénon attractor — capacity dimension

- Computing capacity dimension is a difficult numerical task even for a relatively simple system such as the Hénon attractor.
- Here are some box-counting numbers for the case $a = 1.4$, $b = 0.3$. The first column gives the length of the sample orbit, the other columns give the number of squares of given size required to cover all points on it. The dimension ϵ heads each of these columns.

Length	0.016	0.008	0.004	0.002	0.001	0.0005	0.00025
10^5	810	1870	4336	9793	21856	41335	62672
$2 \cdot 10^5$	814	1879	4385	10030	23593	51164	91944
$5 \cdot 10^5$	814	1887	4417	10175	24409	57832	124586
10^6	814	1890	4428	10220	24692	59689	137366
$2 \cdot 10^6$	814	1892	4437	10244	24832	60572	142705
$5 \cdot 10^6$	814	1894	4437	10268	24923	61143	145458
10^7	814	1894	4439	10274	24963	61371	146404
$2 \cdot 10^7$	814	1894	4440	10280	24987	61499	146970
$5 \cdot 10^7$	814	1894	4440	10282	25008	61575	147387
10^8	814	1894	4440	10284	25016	61605	147541

- The box sizes have been chosen by halving ϵ each time. If, therefore, we simply take the last count in each column and compute d_C from the ratio of two adjacent counts with the longest orbit, we get the following estimates:

$$\frac{\ln(1894/814)}{\ln 2} \approx 1.218, \qquad (\epsilon = 0.016, 0.008)$$
$$\frac{\ln(4440/1894)}{\ln 2} \approx 1.229, \qquad (\epsilon = 0.008, 0.004)$$
$$\frac{\ln(10284/4440)}{\ln 2} \approx 1.212, \qquad (\epsilon = 0.004, 0.002)$$
$$\frac{\ln(25016/10284)}{\ln 2} \approx 1.282, \qquad (\epsilon = 0.002, 0.001)$$
$$\frac{\ln(61605/25016)}{\ln 2} \approx 1.300, \qquad (\epsilon = 0.001, 0.0005)$$

$$\frac{\ln(147541/61605)}{\ln 2} \approx 1.260, \qquad (\epsilon = 0.0005, 0.00025)$$

- There is considerable uncertainty as to whether these numbers are converging, even though some of the calculations involve orbits of length 10^8. The problem is that we really want the box count in the infinite limit, and it is clear that convergence is slow.
- I don't want the investigate this problem any further here. We shall have to make do with the fact that the dimension is obviously not an integer, and that its value, as exhibited by our data, is $d_C \approx 1.28$.

Lyapunov dimension for two-dimensional maps

- On a periodic attractor, the Lyapunov exponents L_1 and L_2 are both negative and all nearby initial points are attracted to the one point. On a strange attractor $L_1 > 0$ and $L_2 < 0$. This is just the information that the attractor is compressed in one direction and expanded in the other by average factors $\lambda_i = \exp(L_i)$ per iteration. One can make a simple argument from which to define a dimension from these exponents, using the property of self-similarity.
- Let $N(\epsilon)$ be the number of squares of side ϵ required to cover the attractor. After one iteration these squares have undergone an uneven change of scale. Choose the orientation of each small square so that one side is stretched and the other shrunk, to give a rectangle of aspect ratio λ_1/λ_2. One iteration of the map replaces the old covering of N squares by a new covering of $(\lambda_1/\lambda_2)N$ squares. These new squares are smaller, having edge size $\lambda_2\epsilon$. Using these two observations,

$$(\lambda_1/\lambda_2)N(\epsilon) \approx N(\lambda_2\epsilon).$$

- Assume, as usual in making definitions of fractal dimension, that

$$N(\epsilon) \approx K\epsilon^{-d_L}.$$

Substituting this formula into the previous relationship between $N(\epsilon)$ and $N(\lambda_2\epsilon)$ gives

$$\ln\lambda_1 - \ln\lambda_2 + \ln K - d_L \ln\epsilon = \ln K - d_L(\ln\epsilon + \ln\lambda_2).$$

Solving for d_L and using the fact that $L_i = \ln(\lambda_i)$, the result is

$$d_L = 1 - L_1/L_2.$$

This leads to the following definition:

- **Lyapunov dimension for a two dimensional map:** The Lyapunov dimension of a strange attractor of a two-dimensional map is given by the formula

$$d_L = 1 - L_1/L_2,$$

provided the Lyapunov exponent L_1 is positive, the other exponent L_2 negative.

- For example, for the Henon map with $a = 1.4$, $b = 0.3$, simple numerical computation involving orbits whose length is only $10^4 \sim 10^6$ gives

$$L_1 \approx 0.39, \qquad L_2 \approx -1.59, \qquad d_L \approx 1 + 0.39/1.59 \approx 1.25.$$

Apart from the fact that we do not require excessive computation to get good estimates, there is the advantage that this does not require repeated computations for a range of values of a size parameter ϵ.

- The question of the relation between the Lyapunov dimension and some of the other fractal dimensions has been the subject of much investigation. In certain special circumstances $d_L = d_C$, but that is not the case for the Hénon attractor, although the two results are close.

Rössler attractor

- I want to finish this lecture by looking at how attractors get their fractal structure. This is more readily seen in the behaviour of non-linear differential equations, the final topic of this course.

- For now, let's look at Rössler equations, which rated a brief mention in an assignment involving *return maps*. They are

$$\frac{dx}{dt} = -y - z,$$
$$\frac{dy}{dt} = x + \alpha y,$$
$$\frac{dz}{dt} = \alpha + z(x - \mu),$$

- Like the Hénon map they defy complete mathematical analysis. A common feature is a chaotic attractor, which can be visualised as the dynamics as a two-dimensional map.
- The solution circles the z-axis in a counter-clockwise direction. This circulating motion suggests a particularly convenient place to take a Poincaré section. (Overhead_19_1).
- Any vertical half plane, radiating out from the z-axis, is pierced precisely once per circuit. Within this section, suitable coordinates are r (distance from the z-axis) and z.
- One circuit of the orbit uniquely maps an initial point (r, z) to the next point (r', z'), that is, the differential equations determine a two-dimensional map

$$r' = f(r, z), \qquad z' = g(r, z),$$

although it is not possible to write down explicit formulae for f and g.
- The orbit under consideration is chaotic, a claim that can be supported by Fourier analysis and the computation of Lyapunov exponents (for the Poincaré section); the largest exponent is positive. (Overhead_19_2 , $L_1 \approx +0.073$).

The Baker's paradigm

- The reduction of the Rössler equations to a two-dimensional map enables us to see that the mechanism whereby the strange attractor is produced is precisely the stretching and folding which operates for the Hénon system.
- This is shown in Overheads_19_3 to _19_10, which displays the intersection of 10^3 points on the attractor at eight different planes of section, spaced $15°$ apart.
- They were chosen so as to sample that part of the attractor where the stretching and folding is most evident.
- The phenomenon is much better observed, as an animation, using *Chaos for Java*, which allows one to observe how the points in the section change, as the plane of section is rotated about the z axis.
- This mechanism for producing fractals, by repeated stretching and folding, is known as the *Baker's paradigm*.