Lecture 18 — Capacity Dimension

Source material: Chapter 5, pp 157–163

- The definition follows from a consideration of how to place a sufficient number of small non-overlapping boxes so as to contain the whole set, and how the number of boxes required to achieve this increases as their size is decreased.
- Given a set A of points in n-dimensional Euclidean space, let $N(\epsilon)$ be the number of n-dimensional cubes of side ϵ needed to cover (contain) every point of A. Unless the set contains only a finite number of points, it is reasonable to expect that

  $$N(\epsilon) \to \infty, \qquad \epsilon \to 0.$$

  Let's ask the question, how is this limit approached?
- For a finite set of K points, $N(\epsilon) = K$ once ϵ is sufficiently small that each box can contain only one point. This may be expressed as

  $$N(\epsilon) \approx K\epsilon^0, \qquad \epsilon \to 0,$$

- For a smooth curve, we expect that

  $$N(\epsilon) \approx K\epsilon^{-1}. \qquad \epsilon \to 0,$$

- For a smooth surface,

  $$N(\epsilon) \approx K\epsilon^{-2}, \qquad \epsilon \to 0,$$

  by a similar argument.
- The dimension of these objects, 0, 1 and 2, may be read off as the power of ϵ in the limiting behaviour of $N(\epsilon)$.
- **Capacity dimension:** The capacity dimension d_C of a set A is defined by the limit

  $$N(\epsilon) \approx K\epsilon^{-d_C}, \qquad \epsilon \to 0,$$

  provided it is well defined.

- The simplest way to read off a power is to take logarithms. That is, take counts $N_1 = N(\epsilon_1), N_2 = N(\epsilon_2), \cdots, \epsilon_k \to 0$, and calculate

$$d_C(A) = \lim_{k\to\infty} \frac{\ln(N_{k+1}/N_k)}{\ln(\epsilon_k/\epsilon_{k+1})}.$$

Capacity dimension of the Cantor set

- The boxes are closed intervals of length ϵ. The most convenient choice is $\epsilon = 1/3^k$, $k = 0, 1, \cdots$. Let us consider how many such boxes are required to cover I_n, for increasing n.

(i) Obviously 3^k intervals of length $\epsilon = 1/3^k$ are required to cover I_0.

(ii) I_1 can be covered by 1 interval of length $\epsilon = 1$, 2 intervals of length $\epsilon = 1/3$, 6 of length $\epsilon = 1/3^2$, and in general by $2 \cdot 3^{k-1}$ intervals of length $\epsilon = 1/3^k$. These latter conclusions arise from the fact that we must cover two subintervals, each of length 1/3.

(iii) I_2 can be covered by 1 interval of length 1, 2 intervals of length $\epsilon = 1/3$, 4 of length $\epsilon = 1/3^2$, and in general by $4 \cdot 3^{k-2}$ intervals of length $\epsilon = 1/3^k$ if $k \geq 2$.

(iv) The general situation is that for I_n, which consists of 2^n subintervals of length $1/3^n$, we require $2^n \cdot 3^{k-n}$ intervals of length $\epsilon = 1/3^k$ if $k \geq n$. However, if $k < n$ we need 2^k.

- Now the Cantor set is defined by taking the limit $n \to \infty$. The other integer k which enters our calculation is used to control the length of the covering intervals. Therefore, the box count needed for a capacity dimension calculation is

$$N(\epsilon_k) = 2^k, \qquad \epsilon_k = 1/3^k.$$

Using the earlier formula to compute d_C from ratios of successive N_k and ϵ_k gives

$$d_C = \frac{\ln(N_{k+1}/N_k)}{\ln(\epsilon_k/\epsilon_{k+1})} = \frac{\ln 2}{\ln 3} \approx 0.631.$$

Even though the Cantor set has zero length, it has a non-zero fractal dimension.

Using self-similarity

- Let $N_{n,k}$ be the number of boxes of size $\epsilon_k = 1/3^k$ required to cover I_n. For fixed k the number reaches a stable value N_k for large n, which is the limit of interest.
- From the scaling hypothesis on which the definition of capacity dimension rests, this expect
$$N_k \approx K\epsilon_k^{-d_C}, \qquad k \to \infty.$$
- The iteration produces I_{n+1} as two copies of I_n, each reduced by a linear factor of 1/3. Covering I_{n+1} with intervals of size ϵ_{k+1} is equivalent to covering each of the original copies of I_n from which it came by intervals of size $3\epsilon_{k+1} = \epsilon_k$. This gives
$$N_{k+1} = 2N_k.$$
- Using the scaling limit for N_k in this equation gives
$$K\epsilon_k^{-d_C} \approx 2K(3\epsilon_k)^{-d_C} \qquad \epsilon_k \to 0.$$
Dependence on the unknown amplitude K immediately cancels, as does dependence on ϵ_k, and we are left with the relation $3^{d_C} = 2$, which gives the capacity dimension.

An asymmetric Cantor set

- Start with the interval $I_0 = [0, 1]$ as before, but this time construct I_{n+1} from two copies of I_n, one scaled by the factor 1/2 and placed at the left, the other scaled by the factor 1/4 and placed at the right. Explicitly, the first few sets are
$$\begin{aligned}
&I_0 = [0, 1],\\
&I_1 = [0, 1/2] \cup [3/4, 1],\\
&I_2 = [0, 1/4] \cup [3/8, 1/2] \cup [3/4, 7/8] \cup [15/16, 1],\\
&I_3 = [0, 1/8] \cup [3/16, 1/4] \cup [3/8, 7/16] \cup [15/32, 1/2]\\
&\qquad \cup [3/4, 13/16] \cup [27/32, 7/8] \cup [15/16, 31/32] \cup [63/64, 1],\\
&\quad \vdots
\end{aligned}$$

- To carry out the box counting it is convenient to choose intervals of size

  $$\epsilon_k = 1/2^k, \qquad k = 0, 1, \cdots$$

  Simple counting gives the following results for $N_{n,k}$ for small n and k:

  $$\begin{array}{lllll} (k=0) & N_{n,0}=1, & n \geq 0, & & \\ (k=1) & N_{n,1}=2, & n \geq 0, & & \\ (k=2) & N_{0,2}=4, & N_{n,2}=3, & n \geq 1, & \\ (k=3) & N_{0,3}=8, & N_{0,3}=6, & N_{n,3}=5, & n \geq 2. \end{array}$$

- Our interest is in the limiting value for $n \to \infty$. These begin as the sequence

  $$N_0 = 1, \qquad N_1 = 2, \qquad N_2 = 3, \qquad N_3 = 5, \qquad \cdots$$

  which is the beginning of a *Fibonacci sequence*, satisfying

  $$N_{k+2} = N_{k+1} + N_k, \qquad k = 0, 1, \cdots$$

  This last observation is a simple manifestation of scaling.

- An exact formula may be given for the numbers in the Fibonacci sequence, but we require only the scaling limit. Substituting

  $$N_k \approx N\alpha^k,$$

  we get $\alpha^{k+2} \approx \alpha^{k+1} + \alpha^k$.

- Rearranging and cancelling the common factor α^k gives the quadratic equation

  $$\alpha^2 - \alpha - 1 = 0,$$

  which has one positive solution

  $$\alpha = \frac{1+\sqrt{5}}{2}.$$

- The capacity dimension may now be computed by replacing N_k and ϵ_k in the equation $N_k \approx \epsilon_k^{-d_C}$ by $N\alpha^k$ and $1/2^k$, respectively. This gives $\alpha^k \approx (1/2^k)^{-d_C}$. Taking logarithms,

  $$k \ln \alpha = d_C \cdot k \ln 2,$$

from which

$$d_C = \frac{\ln(1+\sqrt{5}) - \ln 2}{\ln 2} \approx 0.694.$$

Using scaling directly

- We cannot expect to have exact results for box-counts except in special circumstances.
- What is easy to implement is the scaling argument. Since the iteration uses two copies of the original object to produce the new one, scaling one copy by the factor 1/2 and the other copy by 1/4, we expect the relation

  $$N(\epsilon) \approx N(2\epsilon) + N(4\epsilon), \qquad \epsilon \to 0.$$

- Substituting the scaling limit directly gives the equation

  $$K\epsilon^{-d_C} = K(2\epsilon)^{-d_C} + K(4\epsilon)^{-d_C}.$$

  Removing the dependence on K and ϵ,

  $$1 = 2^{-d_C} + 4^{-d_C}.$$

- This is precisely the quadratic equation we already derived (for the value of 2^{d_C}), so we arrive at the capacity dimension rather quickly this way.