Lecture 17 — Fractals

Source material: Chapter 5, pp 147–154

To reproduce overheads shown in lectures, download the corresponding files from the website and open them with "Chaos for Java"

- Strange attractors are complicated *fractals* with non-integer dimension. For example, the Hénon attractor with $a = 1.4$, $b = 0.3$ has $d \approx 1.28$.
- The term fractal was coined by Mandelbrot. I want to look at some simple examples, and their connection with dynamical systems.
- It is generally accepted that fractals exhibit

(i) Fine structure — there is always more detail to be seen at arbitrarily small scales.

(ii) They are too irregular to be described using traditional geometry, in particular a fractal is not simply a finite collection of smooth geometrical objects.

(iii) Some form of exact or approximate self-similarity.

(iv) The fractal dimension is not the natural geometric dimension.

Hénon attractor

- First some overheads of part of the attractor. Each of the two pictures consists of groups of *thick lines*; the self similarity is evident. (Overhead_17_1 & Overhead_17_2)
- There are simple geometrical trapping regions, for example quadrilaterals, which are mapped into themselves under the Hénon map.
- Denote such a quadrilateral by $\mathcal{Q}$. By any reasonable definition it is two-dimensional and contains an uncountably infinite number of points. Denote also the Hénon map by H; it is a smooth function of the coordinates (x, y), so the image $\mathcal{Q}_1 = H(\mathcal{Q})$ is also a two-dimensional region of bounded by four smooth curves, albeit not straight lines.
- This argument may be applied repeatedly

$$\mathcal{Q} \xrightarrow{T} \mathcal{Q}_1 \xrightarrow{T} \mathcal{Q}_2 \cdots \xrightarrow{T} \mathcal{Q}_n \xrightarrow{T} \cdots$$

Moreover,

$$\mathcal{Q} \supset \mathcal{Q}_1 \supset \mathcal{Q}_2 \cdots \supset \mathcal{Q}_n \supset \cdots$$

- Because of the area contraction, the regions $\mathcal{Q}_n$ decrease geometrically to zero (in area); due to the strong non-linearity, they also increase rapidly in complexity.
- The attractor is defined as the infinite limit

$$\mathcal{Q}_H = \lim_{n\to\infty} \mathcal{Q}_n = \bigcap_{n=0}^{\infty} \mathcal{Q}_n,$$

- $\mathcal{Q}_H$ is a fractal, constructed by a simple iterative procedure.

Cantor Set

- A classic example of a fractal. It is a subset of an interval produced by iterated deletions, each of which removes further pieces.
- For the classic example $I_0 = [0, 1]$ is the closed unit interval. I_1 is obtained from I_0 by removing the open middle third $(1/3, 2/3)$. It is the union of two closed intervals:

$$I_1 = [0, 1/3] \cup [2/3, 1].$$

- I_2 is obtained from I_1 by removing the open middle third of each of its component intervals. It is the union of four closed intervals:

$$I_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1].$$

- There is exact self-similarity here. I_1 consists of two copies of I_0, each linearly scaled by a factor 1/3, and a similar relation exists between I_2 and I_1.
- I_n is the union of 2^n closed intervals, produced from I_{n-1} by deleting the open middle third of each of its 2^{n-1} component intervals.
- If we continue just as far as I_{100}, we will have self-similarity over a range $3^{100} \approx 10^{48}$, i.e., 48 orders of magnitude. To all practical purposes this is infinite self-similarity.
- The Cantor set itself is defined as

$$I_C = \lim_{n\to\infty} I_n = \bigcap_{n=0}^{\infty} I_n. \tag{17.1}$$

That is, x is in I_C if and only if it is in I_n for all n.

Length of the Cantor set

- The length of I_n is naturally defined as the sum of the lengths of its parts. There are 2^n intervals each of length $1/3^n$ so

  $$L_n = (2/3)^n.$$

  Thus the only reasonable definition for the length of I_C is zero, since $\lim_{n\to\infty} L_n = 0$.
- I_C has a very complex structure, as I shall show.

Connection with the tent map

- Let's remove restrictions on the state variable x and parameter t.

  $$f(x) = \begin{cases} 2tx, & (x \le 1/2), \\ 2t(1-x), & (x \ge 1/2), \end{cases} \qquad (t > 0).$$

  Now we require only that x and t are real, with t positive.
- Let's consider what happens to iterations which are outside the interval $[0,1]$. If $x > 1$, then $f(x) < 0$, so every orbit which starts from $x_0 > 1$ is mapped immediately to $x_1 < 0$. If $x_k < 0$, then $x_{k+1} = 2tx_k < 0$, and with $2t > 1$, these iterations progress steadily toward $-\infty$.
- Therefore the infinite interval $(-\infty, 0) \cup (0, \infty)$ is at least part of the basin of attraction of infinity, for all $t > 1/2$.
- What about $t > 1$? If $1/2t < x < 1 - 1/2t$, $f(x) > 1$. So the basin of attraction of infinity includes the open interval $(1/2t, 1 - 1/2t)$ taken from the middle of $[0,1]$.
- The remaining two subintervals are each mapped to the whole interval $[0,1]$ under the first iteration. Therefore, at the next iteration, $x_2 > 1$ for all x_1 in the middle open interval, which is equivalent to x_0 being in the middle part of either of the two ends.
- To recover the middle-third Cantor set, choose $t = 3/2$. The interval I_0 is the original interval $[0,1]$. The interval $I_1 = [0,1/3] \cup [2/3,1]$ is exactly the set of initial values for which the first iterate is still in I_0. I_2 is the set of initial values for which the first iterate is in I_1 and the second iterate in I_0, and so on.

- As for the Cantor set, defined by (17.1), an alternative description is that it is the set of orbits of the tent map (with $t > 1$) which do not escape under iteration.
- Since they are all unstable, there is no hope to observe them by simple numerical iteration. Such computed orbits will generally "get away", unless the arithmetic is done natively to base three, rather than the usual binary of modern computer chips.

Representation using base three arithmetic

- Recall lecture 4, where the tent map with $t = 1$ was described using binary arithmetic. This was because $2t = 2$ in this case. We will adapt that argument to the present need.
- Setting $t = 3/2$, the dynamics is given by the formula

$$x_{k+1} = \begin{cases} 3x_k, & x_k \leq 1/2, \\ 3(1 - x_k), & 1/2 \geq x. \end{cases} \tag{17.2}$$

If we work with base 3 arithmetic, there is a very simple way to express this.

- Every number $0 \leq x \leq 1$ has a *base 3 representation*

$$x = 0.d_1 d_2 d_3 \ldots \tag{17.3}$$

where the d_j are either 0, 1 or 2. The meaning is that

$$0.d_1 d_2 d_3 \ldots = d_1/3 + d_1/3^2 + d_1/3^3 + \cdots$$

- Now it is easy to see that, for the map (17.2), a number with the base 3 representation is mapped as follows:

$$f(0.d_1 d_2 d_3 \ldots) = \begin{cases} 0.d_2 d_3 \ldots & (x \leq 1/3) \\ 1.d_2 d_3 \ldots & (1/3 < x < 2/3) \\ 0.\bar{d}_2 \bar{d}_3 \ldots & (x \geq 2/3) \end{cases}$$

where $\bar{d}_j = 2 - d_j$ is the 2's complement.

- Note that $\bar{1} = 1$ Therefore, $f(x)$ is in the interval $[0, 1]$ if $d_1 \neq 1$; out of it if $d_1 = 1$. At the next iteration, the same consideration applies, except that we require $d_2 \neq 1$ to remain in the interval.

- The conclusion is that the middle-third Cantor set consists of all the numbers with base 3 representation of (17.3), where

$$d_j = 0, 2 \qquad \text{but} \qquad d_j \neq 1, \qquad \text{for all} \qquad j.$$

- But this is in 1 : 1 correspondence with the binary representation of the uncountable set $[0, 1]$, showing that it is uncountable. Furthermore, almost all of the orbits which are in the Cantor set are chaotic: that is, they are non-periodic and have positive Lyapunov exponent.