Lecture 16 — Lyapunov exponents, basin boundaries

Source material: Chapter 4, pp 127–135

To reproduce overheads shown in lectures, download the corresponding files from the website and open them with "Chaos for Java"

Mapping a small ellipse

- For one-dimensional maps the Lyapunov exponent is defined by tracking the image of an interval of negligible length. For two-dimensional maps we know that expansion and contraction is non-uniform; for this reason it is necessary to track a small ellipse.
- The image of a small ellipse is another, although the lengths of the axes and their orientation are changed.

The ellipse

- In cartesian coordinates, the general equation for points $(x+\xi, y+\eta)$ on an ellipse centred at (x, y) is

$$A\xi^2 + 2B\xi\eta + C\eta^2 = 1, \qquad AC - B^2 > 0. \tag{16.1}$$

The first equation defines a *conic section*, the second is the requirement that it be an ellipse, rather than a parabola or hyperbola. The quantity $AC - B^2$ is inversely proportional to the area of the ellipse squared.

Transformation of an ellipse

- Consider the set of points on a small ellipse centred at (x, y). Under one iteration the centre of the ellipse maps to (x', y'), and the relative coordinates of points on the ellipse map according to equation (14.1), which expresses ξ', η' in terms of ξ, η.
- Rearranging equation (14.1) to express ξ, η in terms of ξ', η' gives

$$\xi = \eta'/b, \qquad \eta = \xi' - h'(x)\eta'/b.$$

- Substituting this into (16.1) gives the formula and data for the new ellipse:

$$\begin{aligned} 1 &= A\big(\eta'/b\big)^2 + 2B\big(\eta'/b\big)\big(\xi' - h'(x)\eta'/b\big) + C\big(\xi' - h'(x)\eta'/b\big)^2, \\ &= A'\xi'^2 + 2B'\xi'\eta' + C'\eta'^2, \end{aligned}$$

where

$$\begin{aligned} A' &= C, \\ B' &= \big(B - Ch'(x)\big)/b, \\ C' &= \big(A + Ch'(x)^2 - 2Bh'(x)\big)/b^2. \end{aligned}$$

- An elementary calculation gives

$$A'C' - B'^2 = (AC - B^2)/b^2 > 0.$$

  So the image is an ellipse as claimed.

- From the known meaning of the invariant $AC - B^2$, we also get an independent check on area contraction,

$$\delta'^{\max}\delta'^{\min} = |b|\,\delta^{\max}\delta^{\min}.$$

Lyapunov exponents

- The fact that ellipses map to ellipses enables us to usefully extend the definition of the Lyapunov exponent for a one-dimensional map to the definition of a pair of Lyapunov exponents for a two dimensional map.

- **Lyapunov exponents of two-dimensional map:** For a given initial point (x_0, y_0), the Lyapunov exponents $L_1(x_0, y_0)$, $L_2(x_0, y_0)$ of a map are given by the formulae

$$\begin{aligned} L_1(x_0, y_0) &= \lim_{k\to\infty} \frac{1}{k}\left(\lim_{\delta_0\to 0} \ln|\delta_k^{max}/\delta_0|\right), \\ L_2(x_0, y_0) &= \lim_{k\to\infty} \frac{1}{k}\left(\lim_{\delta_0\to 0} \ln|\delta_k^{min}/\delta_0|\right), \end{aligned}$$

  provided the limits exists.

- Here δ_0 is the radius of an initial circle about (x_0, y_0) and the limit $\delta_0 \to 0$ is taken first to ensure that we only deal with a small ellipse at every stage of the computation.

- For a generalised Hénon map, we already know that there is an area contraction by the factor $|b|$ at each iteration. That is, after k iterations, we know that

$$\big(\delta_k^{max}/\delta_0\big)\big(\delta_k^{min}/\delta_0\big) = b^k.$$

Taking logarithms, this informs us that

$$L_1(x_0, y_0) + L_2(x_0, y_0) = \ln |b|.$$

- One consequence of this is that Lyapunov exponents of these two dimensional maps can never take the value $-\infty$; there are no superstable orbits as in the one-dimensional case.
- Lyapunov exponents computed for the Hénon map are shown here. One clearly sees the fact that their sum is $\ln |b|$. (Overheads_16_1 & 16_2)

Chaotic orbits in two-dimensional maps

- A chaotic orbit of a bounded two-dimensional system is one which is not periodic or eventually periodic, and which has a positive Lyapunov exponent.
- Recall that a dynamical system is said to be chaotic when it is in a regime with chaotic orbits.
- Our computations for the Hénon map, together with the Fourier spectra computed earlier, are compelling evidence that it exhibits chaotic behaviour.

Basin boundaries

- What determines the boundaries between basins? An immediate clue may be found in Overhead_16_3 & 16_4, which are for the Hénon map.
- One picture shows the stable period 1 orbit at $a = 0.3$, $b = 0.3$, together with its basin of attraction and the two fixed points of the map.
- The next picture is similar, except that now $a = 1.4$, at which value both fixed points are unstable, the orbit chaotic.
- A black cross ($+$) is a stable fixed point, a light cross ($\times$) is an unstable fixed point. The common feature is that the point (x^*_-, y^*_-) sits on the basin boundary in both cases.

The unstable fixed point

- The treatment of the eigenvalues $\lambda_\pm$ in lecture 14 does not depend on the formula for the values of $x^*_\pm$.

- Therefore, for the other fixed point (x_-^*, y_-^*), equation (14.7) gives the eigenvalue pair

$$\lambda_\pm = |ax_-^*| \pm \sqrt{|ax_-^*|^2 + b},$$

where I have used the fact that $x_-^* < 0$.

- Now it is easy to show (although I won't do it here) that

$$\lambda_+(a) > 1.$$

- From this, and the relation $\lambda_+\lambda_- = -b$,

$$-b < \lambda_-(a) < 0, \qquad (b > 0),$$

$$0 < \lambda_-(a) < |b|, \qquad (b < 0).$$

- The point (x_-^*, y_-^*) is therefore unstable for all values of a, b, since there is always one direction along which iterates are repelled.

- This kind of unstable fixed point, with one stable and one unstable direction, is known as a hyperbolic point or *saddle point*.

- **Hyperbolic fixed point:** A hyperbolic fixed point of a dissipative two-dimensional map is a point for which the eigenvalues satisfy

$$|\lambda_1| > 1, \qquad |\lambda_2| < 1.$$

- The fixed point (x_-^*, y_-^*) is always hyperbolic, the other point (x_+^*, y_+^*), becomes hyperbolic at the period doubling at $a = a_1$.

Stable or unstable?

- The rôle of the hyperbolic point (x_-^*, y_-^*) is quite profound.

- Under iteration of the system, points which are exactly on the boundary remain on the boundary, for otherwise they are inside one of the basins.

- One of these points is (x_-^*, y_-^*) itself, a fixed point, with one attracting direction.

- The conclusion is that this hyperbolic point is in fact an attractor for points exactly on the boundary!

- This is because the direction of instability is involved in taking iterations away from the boundary, implying that the stable direction is along the boundary itself.
- Although it is not possible to track points along the boundary by simple numerical iteration, the boundary is clearly seen (Overhead_16_5 & 16_6), also the difference between positive and negative values of λ_-.

(i) For the first, $b > 0$, $\lambda_- < 0$. The boundary has an infinite number of pieces which go off to infinity. Moreover, points on the boundary must oscillate from side to side of the hyperbolic point as they approach it.

(ii) For the second, $b < 0$, $\lambda_- > 0$. Now the boundary is fractal. Being infinite, the boundary must be infinitely complicated, because it is unchanged by the stretching and folding action of the map. Since iterates do not switch from side to side, it must therefore be infinitely convoluted within a finite region.

Stable and unstable manifolds

- The systems under consideration are discrete, so individual orbits do not move continuously along curves in the x-y plane.
- But, we have clear evidence that there are continuous plane curves associated with particular aspects of the dynamics.
- Basin boundaries are curves with the property that all iterations which originate exactly on the curve stay on it, even though individual orbits only visit a discrete set of points.
- Since the curve in question is associated with the stable eigenvalue of the hyperbolic point through which it passes, it is called a stable manifold.
- The same hyperbolic point has an unstable eigenvalue with an associated unstable manifold.
- These manifolds are easy to describe, hard to find by actual computation, and extremely difficult to investigate using pure theory; this brings us to the edge of what can be attempted in an elementary introduction such as this.
- The concept originates in the work of Poincaré in the 1890's.