Lecture 15 — Periodic orbits

To reproduce overheads shown in lectures, download the corresponding files from the website and open them with "Chaos for Java"

- A period n orbit is a fixed point of the n-fold composition (f_n, g_n). These functions do not have a simple form; nevertheless it is not difficult to extend theory to them.
- Consider two orbits, the first period n, starting from (x_0^*, y_0^*), the other a nearby orbit starting from $(x_0^* + \xi_0, y_0^* + \eta_0)$. At each step, equation (14.2) applies for the evolution of the small differences (ξ_k, η_k).
- Over n steps, which is also a single step of the n-fold composition, this gives

$$\begin{pmatrix} \xi_n \\ \eta_n \end{pmatrix} = M(x_0^*) \begin{pmatrix} \xi_0 \\ \eta_0 \end{pmatrix}, \tag{15.1}$$

  where

$$M(x_0^*) = \begin{pmatrix} h'(x_{n-1}^*) & 1 \\ b & 0 \end{pmatrix} \cdots \begin{pmatrix} h'(x_1^*) & 1 \\ b & 0 \end{pmatrix} \begin{pmatrix} h'(x_0^*) & 1 \\ b & 0 \end{pmatrix}. \tag{15.2}$$

- This should be compared with (5.2), it is in fact the *chain rule for partial differentiation.* Unlike the one-dimensional case, the order of multiplication must be as shown; matrix multiplication is not commutative.

Stability condition

- Equation (15.1) is at the heart of the stability test test in the general case. All that is required is to replace (14.3) by the more general form

$$\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \lambda \begin{pmatrix} \xi \\ \eta \end{pmatrix}. \tag{15.3}$$

  For simplicity, I have omitted the dependence on x_0^*, writing M_{ij} rather than $M_{ij}(x_0^*)$ here.

- The characteristic equation (14.5) for the eigenvalues λ becomes

$$\lambda^2 - T(x_0^*)\lambda + D = 0, \tag{15.4}$$

  where $T(x_0^*)$ and D are the *trace* and *determinant* of $M_{ij}(x_0^*)$, respectively:

$$T(x_0^*) = M_{11} + M_{22}, \qquad D = M_{11}M_{22} - M_{12}M_{21}.$$

- An important property of matrices is that the determinant of a product is equal to the product of the determinants. Since each of the matrices in the product (15.2) has determinant $(-b)$, this gives the simple result

$$D = (-b)^n. \tag{15.5}$$

- Earlier arguments about the eigenvalues being either real, or a complex conjugate pair, still hold; moreover the orbit is stable provided that both of the eigenvalues are smaller than unity in magnitude.
- Furthermore, since it will always be the case that $|D| < 1$ for dissipative maps, only one eigenvalue can attain the value ± 1, in which case both eigenvalues are real.
- Another, most important, property of the test is that the eigenvalues (but not the eigenvectors) of (15.3) do not change if a different starting point is chosen for the product.
- This follows from a standard result of linear algebra *The eigenvalues of a product of square matrices are invariant under cyclic permutation of the product order.*

Period doubling

- Period doubling occurs when one of the eigenvalues of M attains the value $\lambda = -1$. Period n orbits are also fixed points of the $2n$ composition of the map, whose stability is determined by the eigenvalues of M^2; automatically one of these is $\lambda = +1$.
- This is all very reminiscent of the one-dimensional case.
- Graphically, the mechanism is also the same. In Overheads_15_1 to 15_4, the zero curves of (ϕ, ψ) and (ϕ_2, ψ_2) are plotted for the Hénon map with $a = 0.25 < a_1$ and $a = 0.45 > a_1$.
- They should be compared with Overheads_10_1 & 10_2. For the second composition map, neither of these curves is a straight line, but that is the only difference.
- In fact, it is not difficult to show that the occurrence of an eigenvalue -1 for a fixed point of a map implies that the zero curves intersect transversally, whereas for an eigenvalue $+1$, zero curves are both tangent and have equal curvature at their intersection.

- This is the analogue of the result for one-dimensional maps, for which $f'(x^*) = -1$ implies $f_2'(x^*) = 1$ and $f_2''(x^*) = 0$ at $r = r^*$.

The period doubling cascade

- Period doubling cascades to chaos are commonly observed, just as in the one-dimensional case. How can this happen?
- Apart from the obvious complication of having to deal with functions of two variables, and the associated eigenvalue problem, there is a fundamental question to be resolved.
- In the one-dimensional case, the value of $f_n'(x^*)$ decreases continuously from positive to negative values, passing through a superstable orbit, where $f_n'(x^*) = 0$, on the way.
- For the generalised Hénon map, we have shown that there is a constant area reduction by the factor $|b|$ per iteration, from which it follows that neither of the eigenvalues $\lambda_\pm$ can ever be zero.
- Nevertheless, for any reasonable function $h(x)$ we expect that the eigenvalues will vary continuously as a is changed, exactly as in the one-dimensional case.
- The problem is that, as a function of a, $\lambda_+ = 1$ immediately after the period doubled orbit is born, whereas it is λ_- which must attain the value -1 for the next period doubling.

Resolution of the paradox

- To see it, we first write the solution of the characteristic equation (15.4), using the formula for a quadratic and the explicit form of the determinant given in (15.5):

$$\lambda_\pm = \frac{T \pm \sqrt{T^2 - 4(-b)^n}}{2}. \tag{15.6}$$

- Since n is even after any period-doubling, $(-b)^n = |b|^n > 0$, and real eigenvalues depend on the condition $|T| > 2|b|^{n/2}$.
- It is T which must vary continuously from positive to negative values. As it does so, the pair $\lambda_\pm$ first vary continuously from their initial values 1, $|b|^n$, to the common values $|b|^{n/2}$, $|b|^{n/2}$, while remaining real.

- This happens as the square root in (15.6) decreases steadily to zero. Further decrease of T results in a complex conjugate pair of eigenvalues of constant magnitude $|b|^{n/2}$, but steadily changing phase, until they meet again at the common values $-|b|^{n/2}$, $-|b|^{n/2}$.
- As T continues to decrease, the eigenvalues, now real again, move until they attain the values $-|b|^n$, -1, required for the next period doubling.

Feigenbaum scaling

- In general, values of a_n can only be obtained by careful numerical experimentation, which involves searching for the parameter value which gives $\lambda_- = -1$ for one of the fixed points of the 2^n-fold composition map belonging to the period 2^n orbit. Numerical data, obtained using *Chaos for Java*, is given here, for $b = 0.3$.
- The values are seen to converge geometrically, according to the general scheme for period doubling, with Feigenbaum constant $\delta \approx 4.669$, and $a_\infty \approx 1.05805$.

n	bifurcation	a_n	δ	a_∞
1	1 → 2	0.3675		
2	2 → 4	0.9125		
3	4 → 8	1.0258554050	4.8078872	1.055624
4	8 → 16	1.0511256620	4.4857243	1.0583753
5	16 → 32	1.0565637582	4.6468941	1.0580549
6	32 → 64	1.0577308396	4.6595687	1.0580498
7	64 → 128	1.0579808932	4.6673251	1.0580491
8	128 → 256	1.0580344522	4.6687547	1.0580491

Tangent bifurcations and zero curves

- We have already seen a tangent bifurcation of the Hénon map in lecture 13.
- In terms of zero curves, the mechanism is exactly parallel to that discussed in lecture 12 for one-dimensional maps — tangency of the pair of curves, now defined by equations (13.3).
- As an example, I have chosen the period 7 window which was already remarked on earlier.
- Zero curves and fixed points of the seventh composition of the Hénon map are shown in Overheads_15_5 & 15_6 , for values of a just below, and just above, the critical value.