

Lecture 14 — Area contraction and stability

Source material: Chapter 4, pp 113–118

To reproduce overheads shown in lectures, download the corresponding files from the website and open them with “Chaos for Java”

Generalised Hénon map

- In order to give the ensuing discussion wider applicability, I shall consider the *generalised Hénon map*, defined as

$$\begin{aligned} f(x, y) &= h(x) + y, \\ g(x, y) &= bx. \end{aligned} \quad (|b| < 1)$$

- The more general form still retains the essential feature that non-linearity enters through a single one variable function $h(x)$. For the standard Hénon map, $h(x) = 1 - ax^2$.

Image of a small square

- Now let's consider the transformation of a small square region of side $\delta > 0$, with corners at

$$\begin{array}{cc} (x, y + \delta) & (x + \delta, y + \delta) \\ (x, y) & (x + \delta, y) \end{array}$$

which I have laid out so as to indicate their relative positions in the plane, and let

$$\begin{aligned} x' &= f(x, y) = 1 + h(x) + y, \\ y' &= g(x, y) = bx, \end{aligned}$$

be the image of the bottom left corner under one iteration of the map. Assume for the time being that $b > 0$.

- The top left corner $(x, y + \delta)$ maps to

$$\begin{aligned} x'' &= f(x, y + \delta) = h(x) + y + \delta = x' + \delta, \\ y'' &= g(x, y + \delta) = bx = y'. \end{aligned}$$

This has the same vertical coordinate as the image of the bottom left corner, but for the horizontal coordinate we have $x'' > x'$, so it is the bottom right corner of the image.

- For the image of the bottom right corner, write $h(x + \delta) \approx h(x) + \delta h'(x)$, to get

$$\begin{aligned} x''' &= f(x + \delta, y) = h(x + \delta) + y \\ &\approx h(x) + \delta h'(x) + y = x' + \delta h'(x), \\ y''' &= g(x + \delta, y) = b(x + \delta) = y' + b\delta, \end{aligned}$$

so this becomes the top left corner of the image.

- Finally, for the fourth corner,

$$\begin{aligned} x'''' &= f(x + \delta, y + \delta) = h(x + \delta) + y + \delta \\ &\approx h(x) + \delta h'(x) + y + \delta = x' + \delta h'(x) + \delta, \\ y'''' &= g(x + \delta, y + \delta) = b(x + \delta) = y' + b\delta, \end{aligned}$$

so it is mapped to the top right corner of the image.

Area contraction

- The above shows that the image of the square is (approximately) a parallelogram, with bottom of length δ . The perpendicular height is $b\delta$, so the new area is $b\delta^2$. Furthermore the corners of the image have their orientation reversed compared to the original.
- The case $b < 0$ gives exactly the same equations, however there is no reversal of orientation in this case, while the area is transformed to $|b|\delta^2$.
- In both cases there is a 90° rotation, and the area of any simple small region of the x - y plane is multiplied, at each iteration, by the factor $|b|$.

Dissipative maps

- Two-dimensional maps with the *area contracting* property are said to be dissipative.
- We have just shown that the generalised Hénon map is dissipative for all $|b| < 1$. Other examples will appear later.

Stability of fixed points

- A stable fixed point is one which attracts nearby orbits. This is measured by convergence to zero of the distance

$$d = \sqrt{(x - x^*)^2 + (y - y^*)^2}.$$

- Suppose that a point (x, y) maps to the point $(x', y') = (f(x, y), g(x, y))$, not necessarily close to it, let's calculate the linear approximation for the motion of a nearby point $(x + \xi, y + \eta)$. Assuming ξ and η are small,

$$x + \xi \rightarrow h(x + \xi) + (y + \eta) \approx x' + h'(x)\xi + \eta,$$

$$y + \eta \rightarrow b(x + \xi) = y' + b\xi.$$

- From this it is easy to read off the relative position (ξ', η') as

$$\xi' \approx h'(x)\xi + \eta, \quad \eta' = b\xi, \tag{14.1}$$

or, in matrix form,

$$\begin{pmatrix} \xi' \\ \eta' \end{pmatrix} = \begin{pmatrix} h'(x) & 1 \\ b & 0 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}. \tag{14.2}$$

The eigenvalue problem

- In general, the matrix operation (14.2) changes both length and direction of vectors. However, there may be vectors which only change their length. Finding them is an eigenvalue problem.
- Suppose the vector (ξ', η') is a multiple of (ξ, η) , and denote the multiplying factor by λ ; this gives the *eigenvalue equation*

$$\begin{pmatrix} h'(x) & 1 \\ b & 0 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \lambda \begin{pmatrix} \xi \\ \eta \end{pmatrix}. \tag{14.3}$$

- As a pair of homogeneous simultaneous equations, this is

$$\begin{aligned} (h'(x) - \lambda)\xi + \eta &= 0, \\ b\xi - \lambda\eta &= 0. \end{aligned} \tag{14.4}$$

- Eliminating ξ, η gives the *characteristic equation* which must be satisfied by λ ,

$$\lambda^2 - \lambda h'(x) - b = 0. \tag{14.5}$$

This will determine, in general, two distinct *eigenvalues*, λ_{\pm} .

- Once the eigenvalues are determined, the corresponding *eigenvectors* may be obtained by substituting into either of equations (14.4). This gives only the ratio $\xi : \eta$.

Properties of the eigenvalues and eigenvectors

- It is useful to write the quadratic (14.5) in factored form as

$$\lambda^2 - \lambda h'(x) - b = (\lambda - \lambda_+)(\lambda - \lambda_-).$$

- From this it is immediate that

$$\lambda_+ + \lambda_- = h'(x), \quad \lambda_+ \lambda_- = -b. \quad (14.6)$$

The second relation implies that, in the dissipative case ($|b| < 1$) only one of the eigenvalues can have magnitude greater than unity.

- Now let's write a formula for the eigenvalues:

$$\lambda_{\pm} = \frac{h'(x) \pm \sqrt{h'(x)^2 + 4b}}{2}. \quad (14.7)$$

There are three distinct possibilities:

Eigenvalues real but unequal

- In this case, the corresponding eigenvectors are not in the same direction, so they may be used to define coordinate axes. Denote the solutions of (14.4) corresponding to λ_{\pm} by (ξ_{\pm}, η_{\pm}) , respectively.
- An arbitrary initial value (ξ_0, η_0) may be represented in terms of these vectors as

$$\begin{pmatrix} \xi_0 \\ \eta_0 \end{pmatrix} = \alpha_+ \begin{pmatrix} \xi_+ \\ \eta_+ \end{pmatrix} + \alpha_- \begin{pmatrix} \xi_- \\ \eta_- \end{pmatrix}, \quad (14.8)$$

where the components α_{\pm} may be found from ξ_0 and η_0 . So we obtain the general formula

$$\begin{pmatrix} \xi_k \\ \eta_k \end{pmatrix} = \lambda_+^k \alpha_+ \begin{pmatrix} \xi_+ \\ \eta_+ \end{pmatrix} + \lambda_-^k \alpha_- \begin{pmatrix} \xi_- \\ \eta_- \end{pmatrix}.$$

This shows that the fixed points are stable if $|\lambda_{\pm}| < 1$.

- In the event that one of the eigenvalues is larger than unity in magnitude, one of the components grows geometrically, and the fixed point is unstable.

Eigenvalues are complex

- Then the formula (14.7) shows that they are complex conjugate, and satisfy

$$\bar{\lambda}_+ = \lambda_-, \quad |\lambda_{\pm}| = \sqrt{|b|},$$

where $\bar{\lambda}$ denotes the complex conjugate. In this case equation (14.8) becomes

$$\begin{pmatrix} \xi_0 \\ \eta_0 \end{pmatrix} = \alpha_+ \begin{pmatrix} \xi_+ \\ \eta_+ \end{pmatrix} + \bar{\alpha}_+ \begin{pmatrix} \bar{\xi}_+ \\ \bar{\eta}_+ \end{pmatrix}.$$

- Although all the quantities are complex, the complex conjugation which occurs ensures that the sum is real. There are still two independent components, the real and imaginary parts of α_+ . Iteration of the map, to linear approximation, now gives

$$\begin{pmatrix} \xi_k \\ \eta_k \end{pmatrix} = \lambda_+^k \alpha_+ \begin{pmatrix} \xi_+ \\ \eta_+ \end{pmatrix} + \bar{\lambda}_+^k \bar{\alpha}_+ \begin{pmatrix} \bar{\xi}_+ \\ \bar{\eta}_+ \end{pmatrix}.$$

It follows that the iterations converge if $|b| < 1$.

Eigenvalues real and equal

- This occurs as a transitional case between the previous two. Since $|\lambda_{\pm}| = |b|^{1/2} < 1$, the fixed point is stable.

Loss of stability for the Hénon map

- For $a > 0$, $b > 0$, we have $h'(x_+^*) = -2ax_+^*$, from which

$$\lambda_{\pm}(a) = -ax_+^* \pm \sqrt{(ax_+^*)^2 + b}.$$

Both eigenvalues are real since $b > 0$.

- Recall equations (14.6). Substituting the present values into the first gives the condition

$$\lambda_+(a) + \lambda_-(a) = -2ax_+^*. \tag{14.9}$$

- Now x_+^* is a function of a , moreover $x_+^* \geq 0$. Therefore $\lambda_+(a)$ is positive, since $\sqrt{(ax_+^*)^2 + b} > ax_+^*$; also $\lambda_+(a) \leq b^{1/2}$, since $\sqrt{(ax_+^*)^2 + b} < ax_+^* + b^{1/2}$ (try squaring both sides), and $\lambda_-(a)$ is negative. Summarising, as functions of a ,

$$0 < \lambda_+(a) < b^{1/2}, \quad \lambda_-(a) < 0.$$

- This means that the fixed point cannot become unstable through the direction connected with λ_+ ; rather it must be because λ_- passes through the critical value -1 .
- Moreover, iterates whose position relative to (x_+^*, y_+^*) is in the direction connected with λ_- oscillate from side to side. At the critical value of a for which $\lambda_- = -1$, a period doubling bifurcation occurs.

Period doubling

- Let's find the critical value, a_1 , for this first period doubling. Substituting $\lambda_+ = b$, $\lambda_- = -1$ into the condition (14.9) gives

$$b - 1 = -2ax_+^*;$$

substituting further the formula (13.4) for x_+^* leads to the required equation for the critical value of a ,

$$-(1 - b) = (1 - b) - \sqrt{(1 - b)^2 + 4a},$$

- The solution is

$$a_1 = \frac{3}{4}(1 - b)^2.$$

If $b = 0.3$, it yields $a_1 = 0.3675$.

- Finding a formula for the next period doubling is a challenge, although it is clearly seen in bifurcation diagrams that this is the beginning of a period doubling cascade. (Overheads_14_1 & 14_2)