

## Lecture 13 — The Hénon map

*Source material: Chapter 4, pp 107–112*

*To reproduce overheads shown in lectures, download the corresponding files from the website and open them with “Chaos for Java”*

- **Two-dimensional system:** A pair of equations of the form

$$x_{k+1} = f(x_k, y_k; \mu), \quad y_{k+1} = g(x_k, y_k; \mu), \quad (13.1)$$

is called a discrete two-dimensional dynamical system, whose state variable is the vector  $(x_k, y_k)$ . The coefficient(s)  $\mu$  are the control parameter(s).

- Recall for one-dimensional maps we restricted our attention to an interval. For two-dimensional systems we simply assume that  $x_k, y_k$  are real numbers and regard (13.1) as a map from the two-dimensional plane to itself.
- **Orbit:** The sequence of points  $(x_k, y_k)$ ,  $k = 0, 1, \dots$  generated by the system (13.1) is called an orbit of the system, while the point  $(x_0, y_0)$  from which it commences is the initial state. Every orbit is uniquely specified by the initial state  $(x_0, y_0)$ .

### The Hénon map

- The classic example is the two-dimensional Hénon map

$$f(x, y) = 1 - ax^2 + y,$$

$$g(x, y) = bx,$$

with  $a$  and  $b$  as parameters.

- It can be considered as an extension of the logistic map to two dimensions.

### The Attractor of Hénon

- Take  $a = 1.4$  and  $b = 0.3$  and examine the set of points in the  $x$ - $y$  plane which are visited by a single long orbit of the map. (Overhead\_13\_1)
- Recall the discussion of forward limit sets in Lecture 10. Here we are attempting to visualise the set of points which are visited arbitrarily closely, infinitely often, by the orbit.

- By zooming in closer we get a hint of the infinitely complex, fractal, structure of a strange attractor. (Overhead\_13\_2)
- A forward limit set with the property that all orbits which start sufficiently close to it converge to points in it, will be called an **attractor**.
- The strange attractor of the Hénon map contains an uncountably infinite number of points. For any chosen distance  $\delta$ , no matter how small, after some number  $N$  (which will depend on the initial state  $(x_0, y_0)$  as well as on  $\delta$ ) every iteration  $(x_k, y_k)$  of the orbit starting from  $(x_0, y_0)$  is within distance  $\delta$  of some point of the attractor.

### Basins of attraction

- Basin of attraction: The set of initial conditions whose orbits converge to a given attractor constitute its basin of attraction.
- Note that because of the  $ax^2$  term the point at infinity is also an attractor, so there is also a basin of infinity.
- Regardless of whether the attractor is periodic or strange, its basin of attraction may be infinitely complex, even fractal. Some examples:
  - (i)  $a = 1.4$  and  $b = 0.3$  – the orbit is infinitely complex but the basin of attraction (shaded light grey) is relatively simple. (Overhead\_13\_3)
  - (ii)  $a = 1.4$  and  $b = -0.3$  – the orbit is stable period 2 but the basin boundary has become infinitely complex. (Overhead\_13\_4)
- In each case the dark grey areas are the basins of infinity.

### Bifurcations

- We can investigate the sequences  $x_k$  using tools already developed for one-dimensional maps. Since  $y_{k+1} = bx_k$  the dynamical information is completely contained in the sequence  $x_k$  alone.
- The final state diagrams display quite clearly period doubling cascades to chaos and tangent bifurcation. Note also the most prominent periodic window has period 7, unlike the logistic map where period 3 has this distinction. (Overhead\_13\_5)

- Two of the period 7 orbits appear to cross. However, they do not, as they actually have different  $y$  values at the point where the  $x$  values coincide. (Overhead\_13.6)
- As further evidence that there is a tangent bifurcation from chaos to the periodic 7 attractor, consider Fourier spectra at the values  $a = 1.226$ ,  $b = 0.3$  and  $a = 1.227$ ,  $b = 0.3$ , taken just below and above the critical value of  $a$ . They provide strong evidence of aperiodic behaviour which suddenly switches to a stable period 7 orbit. The peaks which prefigure the stable orbit are clearly visible; note that sample size 7000 is chosen because of the importance of its divisibility by the period. (Overheads\_13.7 & 13.8)

### Coexisting attractors

- Looking at  $a = 1.07$  and  $b = 0.3$  with initial state  $(0.5, 0.5)$  we see a strange attractor. However, with initial state  $(-0.8, -0.4)$  there is a period 6 orbit.
- Each orbit has its basin of attraction, and infinity is a third attractor. Clearly the basins of the two bounded attractors, as indicated by the different shades, are intertwined in an extremely complex way. (Overhead\_13.9)

### Fixed points

- **Fixed point** Any pair  $(x^*, y^*)$  for which

$$\begin{aligned} f(x^*, y^*) &= x^*, \\ g(x^*, y^*) &= y^*, \end{aligned} \tag{13.2}$$

is called a fixed point of the two-dimensional dynamical system.

- Define two functions  $\phi(x, y)$ ,  $\psi(x, y)$ , as

$$\begin{aligned} \phi(x, y) &= x - f(x, y), \\ \psi(x, y) &= y - g(x, y). \end{aligned}$$

In terms of these functions, the fixed point equation (13.2) is simply the requirement that  $\phi(x, y)$  and  $\psi(x, y)$  be simultaneously zero. Fixed points may therefore be located as points of intersection of curves in the plane.

- The zero-curves of  $\phi$  and  $\psi$ , for the Hénon map with  $a = 1.4$ ,  $b = 0.3$ , are shown in Overhead\_13.10.

### Exact formula

- For the Hénon map, the functions are simply

$$\begin{aligned}\phi(x, y) &= x - 1 + ax^2 - y, \\ \psi(x, y) &= y - bx,\end{aligned}\tag{13.3}$$

so the corresponding zero-curves are a parabola (if  $a \neq 0$ ) and a straight line. There are two fixed points, provided that the two curves intersect.

### Compositions and periodic orbits

- Compositions of the map are defined by formulae which are easy to define, but difficult to deal with

$$f_2(x, y) = f(f(x, y), g(x, y)), \quad g_2(x, y) = g(f(x, y), g(x, y)),$$

and, in general

$$f_n(x, y) = f_{n-1}(f(x, y), g(x, y)), \quad g_n(x, y) = g_{n-1}(f(x, y), g(x, y)).$$

- **A periodic orbit** is a set of  $n$  fixed points of the  $n$ -fold composition, each of which is not a fixed point of an  $m$ -fold composition for any  $m < n$ , which are visited in sequence under iteration of the map.

- If we define functions  $\phi_n(x, y)$ ,  $\psi_n(x, y)$ , by

$$\begin{aligned}\phi_n(x, y) &= x - f_n(x, y), \\ \psi_n(x, y) &= y - g_n(x, y),\end{aligned}$$

then their zeros are (possibly complicated) curves in the  $x$ - $y$  plane, whose intersections are the fixed points of the  $n$ -fold composition map. The properties of these intersections therefore determine the structure of the periodic orbits, stable and unstable, of the map.

- I show the zero-curves of  $\phi_n$  and  $\psi_n$ , for the second and fourth compositions of the Hénon map, with  $a = 1.4$  and  $b = 0.3$ . Fixed points are marked by a cross. (Overheads\_13.11 & 13.12)
- There are four fixed points with  $n = 2$ , two of which are the fixed points for  $n = 1$ . The new pair constitute a single period 2 orbit.

- There are eight fixed points with  $n = 4$ , of which two are fixed points of the map, and two more belong to the period 2 orbit. This leaves four fixed points which constitute a single period 4 orbit.
- A catalogue of periodic orbits may be constructed, exactly as on page The data for the Hénon map is extended to  $n = 8$  here:

$n$	fixed points	period $m < n$	new points	period $n$
1	2	—	2	2
2	4	2	2	1
3	2	2	—	—
4	8	4	4	1
5	2	2	—	—
6	16	4	12	2
7	30	2	28	4
8	64	8	56	7

### Formulae for the Hénon map

- Using the second of equations (13.3) to eliminate  $y$  from the first, gives a quadratic expression for  $x^*$ :

$$ax^{*2} + (1 - b)x^* - 1 = 0.$$

Remembering that  $y^* = bx^*$ , there are two solutions  $(x_{\pm}^*, y_{\pm}^*)$ , namely

$$x_{\pm}^* = \frac{-(1 - b) \pm \sqrt{(1 - b)^2 + 4a}}{2a}, \quad y_{\pm}^* = bx_{\pm}^*. \quad (13.4)$$

Provided that  $a > 0$ , they satisfy  $x_+^* > 0$ ,  $x_-^* < 0$

- Obviously they exist only for  $(1 - b)^2 + 4a \geq 0$ , or

$$a \geq a_0 = -\frac{1}{4}(1 - b)^2.$$

If  $b = 0.3$ , then  $a_0 = -0.1225$ . The birth of a pair of fixed points at a critical parameter value is typical of a tangent bifurcation. (Overheads\_13\_13)