Lecture 12 — Tangent bifurcations

Source material: Chapter 3, pp 89–95

To reproduce overheads shown in lectures, download the corresponding files from the website and open them with "Chaos for Java"

- Periodic windows are a common feature of final state diagrams. For example, in the logistic map at $r = 3.96$, the chaotic behaviour suddenly switches to a stable period 4 window, which takes the period doubling route back to chaos. (Overhead_12_1)
- The mechanism which gives birth to this period 4 orbit is seen quite clearly in the graphs of f_4.(Overhead_12_2)
- At $r = 3.96$ there are eight unstable fixed points of f_4. The function has three minima and one maximum which do not quite meet the line $y = x$. With a slight increase in r, the four peaks push through the line, creating eight new fixed points. Initially four are stable, four unstable, so the collision gives birth to a pair of period 4 orbits, one stable and one unstable. (Overheads_12_3 & 12_4)
- At the critical value $r = r^*$ when the collision takes place, the line $y = x$ is tangent to the graph – a *tangent bifurcation*.

Approximate description

- We already looked at the theory in lecture 8. For convenience I repeat it here with a little more detail.
- If $r = r^*$ and $x = x^*$, at a point of tangency, then $\phi_n(x) = x - f_n(x) = 0$ and $\phi'_n(x^*) = 0$, so

$$\phi_n(x) \approx \frac{\phi''_n(x^*)}{2}(x - x^*)^2 = K(x - x^*)^2.$$

- Note that ϕ_n has a minimum at $x = x^*$ if $\phi''_n(x^*) > 0$, or a maximum if $\phi''_n(x^*) < 0$.
- To unfold the bifurcation, set $r = r^* + \delta r$ and $\phi_n(x^*) \approx A\delta r$, for some constant A. Then the approximated fixed point equation is

$$A\delta r + K(x - x^*)^2 = 0,$$

a sideways parabola.

- If A and K have the same sign, then there is no solution of this equation for $r < r^*$, while for $r > r^*$ the solution is

$$x^*_\pm \approx x^* \pm \sqrt{(A\delta r)/K}.$$

- This leads to a *periodic window.*
- The reverse situation, that A and K have opposite sign, causing a periodic orbit to suddenly loose its stability by joining up with an unstable orbit of the same period, does not occur for the logistic map. It is known as a *reverse tangent bifurcation.*
- I show examples of tangent bifurcations for the period three and period four windows of the logistic map. Note the many similarities. (Overheads_12_5 & 12_6)

Period 3 for the logistic map

- The critical value r^* for the period 3 tangent bifurcation of the logistic map can be found exactly.
- Now $\phi_3(x) = x - f_3(x)$ is an 8th order polynomial in x. Since the two fixed points of f are also fixed points of f_3,

$$\phi_3(x) = x(1 - r + rx)P_6(x),$$

where now $P_6(x)$ is sixth order in x.

- However, $P_6(x)$ can't be factored for arbitrary r. But at $r = 1 + \sqrt{8}$, P_6 does factor as the square of a cubic, so it has three double roots in this case. The exact period 3 tangent bifurcation is thus seen to be at

$$r^* = 1 + \sqrt{8} = 3.828427\ldots$$

Sarkovskii's theorem — infinitely many periodic windows

- It can be shown that if f is a unimodal map with a period p orbit, then there is a period q orbit for every integer q which precedes p (denoted by $q \leftarrow \cdots \leftarrow p$) in the sequence:

$$\begin{array}{ccccccccccc}
1 & \leftarrow & 2 & \leftarrow & 2^2 & \leftarrow & \cdots & \leftarrow & 2^n & \cdots & \\
\cdots & \leftarrow & 2^m\cdot 9 & \leftarrow & 2^m\cdot 7 & \leftarrow & 2^m\cdot 5 & \leftarrow & 2^m\cdot 3 & \leftarrow & \\
\cdots & \leftarrow & 2^2\cdot 9 & \leftarrow & 2^2\cdot 7 & \leftarrow & 2^2\cdot 5 & \leftarrow & 2^2\cdot 3 & \leftarrow & \\
\cdots & \leftarrow & 2\cdot 9 & \leftarrow & 2\cdot 7 & \leftarrow & 2\cdot 5 & \leftarrow & 2\cdot 3 & \leftarrow & \\
\cdots & \leftarrow & 9 & \leftarrow & 7 & \leftarrow & 5 & \leftarrow & 3. & &
\end{array}$$

- Several points are to be noted:

(i) 1 and 3 are at the ends of the sequence; all other positive integers fall between them in Sarkovskii's ordering.

(ii) The existence of, for example, a period 8 orbit implies the existence of at least one period 4 orbit, at least one period 2 orbit, and at least one fixed point.

(iii) The existence of a period 5 orbit may not imply the existence of a period 3 orbit, but it does imply orbits of all the other periods.

(iv) The existence of a period 3 orbit implies orbits of every period, whence the phrase "Period three implies chaos".

- The theorem tells us nothing about the stability of the orbits, or the parameter values for which they may be observed.
- But it does tell us to expect infinite complexity (Overheads_12_7 & 12_8)

Intermittent behaviour

- Look at the iterates of the logistic map just before the period 3 tangent bifurcation, i.e. just before $r^* = 1 + \sqrt{8}$. In particular, when the iterations are joined so as to show the action of the third composition map. That is, $x_0, x_3, x_6, \cdots$ are joined by straight lines, similarly for $x_1, x_4, x_7, \cdots$ and $x_2, x_5, x_8, \cdots$.
- We see long stretches of almost stable period 3, called *laminar regions*, interspersed by *chaotic bursts*. Notice that the laminar regions are longer for $r = 3.8284$, which is closer to $1 + \sqrt{8}$, than for $r = 3.8283$. (Overheads_12_9 & 12_10)
- The influence of the laminar regions, which prefigure the period 3 orbit, is seen quite clearly in the accompanying Fourier spectra. A sharp peak is developing at frequency 1/3. (Overhead2_12_11 & 12_12)
- Looking at the cobweb plot, it is clear that the long stretches of almost periodic behaviour are caused by the iterates getting trapped in the narrow channel near the minimum of $\phi_3(x)$. (Overhead_12_13)

A scaling relation

- I shall investigate what is going on in the laminar region by considering the cobweb plot — there is a narrow channel near the minimum of $\phi_3(x)$.
- Let's model the situation using the approximation derived earlier, which gives

$$\phi_3(x) \approx A\Delta r + K(x-a)^2.$$

It is clear that the progress made in a single iteration is $\Delta x_k = (x_{k+1} - x_k) = \phi_3(x_k)$.

- Since the number of iterations required to get through the channel is large, we can think of the rate of progress as if the iteration count k is a continuous quantity, just as we think of large numbers of insects in a population model.
- This gives

$$\frac{dx}{dk} \approx A\Delta r + K(x-a)^2.$$

Taking the reciprocal,

$$\frac{dk}{dx} \approx \frac{1}{A\Delta r + K(x-a)^2},$$

from which we can obtain an estimate of the number of iterations to progress from $x = a - \Delta x$ to $x = a + \Delta x$, as

$$k(a+\Delta x) - k(a-\Delta x) \approx \int_{a-\Delta x}^{a+\Delta x} \frac{dx}{A\Delta r + K(x-a)^2}.$$

- When Δr is small compared with Δx, which is the present case, we might as well replace $\pm\Delta x$ by $\pm\infty$, after which our estimate of the total number L of iterations required to navigate the bottleneck becomes

$$L \approx \int_{-\infty}^{\infty} \frac{dx}{A\Delta r + K(x-a)^2} = \frac{\pi}{\sqrt{AK\Delta r}}.$$

- This is a standard integral, but I omit the details because the only thing which matters is the dependence on Δr, which can be found without actually evaluating the integral. So we have the scaling relation between the length of a typical laminar region, L, and the distance from the critical parameter value, δr:

$$L \sim M(\delta r)^{-1/2}.$$

The relation is readily observed in numerical experiments.