Lecture 11 — Period doubling route to chaos

Source material: Chapter 3, pp 79–87

To reproduce overheads shown in lectures, download the corresponding files from the website and open them with "Chaos for Java"

Schwarzian magic

- In the last lecture, I developed a simple and general theory which explains period doubling cascades, caused by successive derivatives, $f'_n, f'_{2n}, f'_{2^2n}, \cdots$ passing through the critical value -1.
- It depends on knowing the sign of the third derivatives, evaluated at points we can't even know as a general formula.
- A little magic comes to the rescue. Let's look at the formula for $f_2'''(x)$

$$f_2'''(x) = \frac{d}{dx}\left[f''(f(x)) \cdot f'(x)^2 + f'(f(x)) \cdot f''(x)\right]$$
$$= f'''(f(x)) \cdot f'(x)^3 + 3f''(f(x)) \cdot f''(x) \cdot f'(x) + f'(f(x)) \cdot f'''(x).$$

  It involves all three derivatives f', f'' and f''', but at $r = r^*$ we know that $f'(x^*) = \pm 1$ and $f''(x^*) = 0$, so the second derivatives make no contribution at x^*. That is,

$$f_2'''(x^*) = 2f'(x^*)f'''(x^*);$$

  some memory of the third derivative survives the period doubling.

- To exploit this fact, consider a rather complicated function $S[f]$ constructed using the following combination of the derivatives of f:

$$S[f] = 2f'f''' - 3f''^2. \tag{11.1}$$

(i) It has the property that, if we know the sign of $S[f]$ at a point where $f' = -1$ and $f'' = 0$, then we know the sign of f_2''' at that point.

(ii) $S[f]$ is related to the *Schwarzian derivative* — in fact equation (11.1) must be divided by $2f'^2$ to get the usual definition.

- Consider what happens under function composition. To avoid confusion, consider first the most general composition

$$h(x) = f(g(x)).$$

- Then

$$S[h](x) = S[f](g(x)) \cdot g'(x)^4 + S[g](x) \cdot f'(g(x))^2. \tag{11.2}$$

This is a general formula, true for all x.

- We don't care about the actual values, we simply want to know whether or not $S[h](x)$ is negative. Equation (11.2) shows that

$$\text{if} \qquad S[f](x) < 0 \qquad \text{and} \qquad S[g](x) < 0, \qquad \text{then} \qquad S[h](x) < 0.$$

Cascades to Chaos

- The period doubling cascade proceeds at an ever more frenetic pace as the parameter increases. (Overheads_11_1 & 11_2)
- Define r_n as the value of r at the point where the nth period doubling occurs. We adopt a *scaling hypothesis* that the sequence of values r_n converges geometrically to a limiting value r_∞, that is

$$r_n - r_\infty \approx A\delta^{-n}, \qquad n \to \infty.$$

Numerical estimates

- Taking the difference of two adjacent equations,

$$(r_n - r_\infty) - (r_{n+1} - r_\infty) = r_n - r_{n+1} \approx A\delta^{-n}(1 - \delta^{-1}).$$

The ratio of two such relations gives

$$\frac{r_n - r_{n+1}}{r_{n+1} - r_{n+2}} \approx \delta.$$

- To estimate r_∞ , write

$$0 = (r_n - r_\infty) - \delta(r_{n+1} - r_\infty).$$

Solving for r_∞ gives

$$r_\infty \approx \frac{r_n r_{n+2} - r_{n+1}^2}{r_n - 2r_{n+1} + r_{n+2}}$$

Scaling relations

- A *scaling relation* takes a form such as

$$A_n \approx A_\infty \alpha^n, \qquad n \to \infty, \qquad \text{or} \qquad A(\epsilon) \approx A(0)\epsilon^{-d}, \qquad \epsilon \to 0.$$

We shall encounter both forms in these lectures.

- The relation $r_n - r_\infty \approx A\delta^{-n}$ as $n \to \infty$ has precise meaning

$$\lim_{n\to\infty} \frac{r_n - r_\infty}{\delta^{-n}} = A.$$

Here A is a constant depending on the particular map. However, δ is a *universal constant* which can be the same for different maps.

Logistic Map

- For the main period doubling sequence of the logistic map the first few r_n values are

$$\begin{array}{lll} r_1 = 3, & r_2 = 3.4494897428, & r_3 = 3.5440903596, \\ r_4 = 3.5644072661, & r_5 = 3.5687594195, & r_6 = 3.5696916098, \quad \cdots \end{array}$$

They were obtained using *Chaos for Java* to locate, to ten decimal places, the values of r at which the derivative $f_n'(x_i^*)$ attains the critical value -1.

- Since we need three adjoining values of r_n for each estimate, this gives the following sequence of numerical results:

$$\begin{array}{lll} (r_1, r_2, r_3): & \delta \approx 4.7514462, & r_\infty \approx 3.5693075. \\ (r_2, r_3, r_4): & \delta \approx 4.6562512, & r_\infty \approx 3.5699641. \\ (r_3, r_4, r_5): & \delta \approx 4.6682415, & r_\infty \approx 3.5699458. \\ (r_4, r_5, r_6): & \delta \approx 4.6687414, & r_\infty \approx 3.5699440. \end{array}$$

- We take the obvious convergence as strong evidence of the scaling hypothesis.

- We can also examine how distances d_n between points on period 2^n orbits scale with increasing n. It will be seen that they also scale geometrically, this time to zero,

$$d_n \approx B\alpha^{-n}, \qquad n \to \infty.$$

- Theory and experiment both give the value

$$\alpha \approx 2.502908\ldots$$

- Both α and δ are *universal constants*, known as Feigenbaum's constants. They are the same for all period doubling cascades of a smooth unimodal map with a quadratic maximum.

Superstable orbits

- Recall that stability is determined by the derivative. Parameter values for which the derivative is zero are special.
- Normally convergence to stable points is *linear*:

$$|x_{i+n} - x_i^*| \approx |f_n'| \cdot |x_i - x_i^*|,$$

If $f_n' = 0$, it can be shown that the convergence is quadratic:

$$|x_{i+n} - x_i^*| \approx |f_n''/2| \cdot |x_i - x_i^*|^2.$$

- **Superstable:** A superstable period n orbit is one for which the derivative f_n' has the values $f_n'(x_i^*) = 0$, somewhere between where $f_n'(x^*) = +1$ and $f_n'(x^*) = -1$.
- Denote the values of r at which the period 2^n orbit is superstable by $\bar{r}_n$. The Lyapunov exponent has the value $-\infty$ wherever a derivative used in its computation becomes zero, as seen in graphs. (Overheads_11_3 & 11_4)

Logistic map

- We already know two superstable orbits for the logistic map: $\bar{r}_0 = 2$ and for $\bar{r}_1 = 1+\sqrt{5}$. Numerical values for the first few $\bar{r}_n$ are

$$\begin{array}{llll} \bar{r}_0 = 2, & \bar{r}_1 = 3.2360679775, & \bar{r}_2 = 3.4985616993, & \\ \bar{r}_3 = 3.5546408628, & \bar{r}_4 = 3.5666673799, & \bar{r}_5 = 3.5692435316, & \cdots \end{array}$$

- Clearly the parameter values $\bar{r}_n$ should scale with the same universal Feigenbaum constant δ, according to the law

$$\bar{r}_n - r_\infty \approx \bar{A}\delta^{-n}, \qquad n \to \infty.$$

Using the above data,

$$\begin{array}{lll} (\bar r_0,\bar r_1,\bar r_2): & \delta\approx 4.7089430, & r_\infty\approx 3.5693349. \\ (\bar r_1,\bar r_2,\bar r_3): & \delta\approx 4.6807710, & r_\infty\approx 3.5698766. \\ (\bar r_2,\bar r_3,\bar r_4): & \delta\approx 4.6629596, & r_\infty\approx 3.5699507. \\ (\bar r_3,\bar r_4,\bar r_5): & \delta\approx 4.6684035, & r_\infty\approx 3.5699458. \end{array}$$

Scaling and self-similarity

- Using the formula for the derivative in the condition for superstability,

$$f_n'(x_i) = \prod_{j=0}^{n-1} f'(x_{i+j}) = 0,$$

we see that, for a smooth unimodal map, an orbit is superstable if and only if $f'(x_k^*) = 0$ for some point x_k^* on it. This means that the maximum point $x_{\max}$ of f must belong to the orbit.

- Suppose we trace the branch of the bifurcation diagram which passes through the point $(\bar r_n, x_{\max})$ back to the point r_n at which this period 2^n orbit came in existence and the forward along the other branch. The two points are related by the composition $f_{2^{n-1}}$.
- Denote by d_n the distance between $x_{\max}$ and $f_{2^{n-1}}(x_{\max})$, that is

$$f_{2^{n-1}}(x_{\max}) = x_{=rmmax} + d_n, \qquad f_{2^{n-1}}(x_{\max} + d_n) = x_{\max}$$

- Overheads_11_5 & 11_6 illustrate what we are doing, also the self-similarity.
- Corresponding to the values $\bar r_n$ already found for the logistic map, the d_n values, and the ratios d_n/d_{n+1}, are

$$\begin{array}{ll} d_1 = 0.30901699, & \\ d_2 = 0.11640177, & d_1/d_2 = 2.65475, \\ d_3 = 0.04597521, & d_2/d_3 = 2.53184, \\ d_4 = 0.01832618, & d_3/d_4 = 2.50872, \\ d_5 = 0.00731843, & d_4/d_5 = 2.50411, \end{array}$$

in good agreement for such a limited data set.