Lecture 10 — Period doubling

Source material: Chapter 3, pp 71–77

To reproduce overheads shown in lectures, download the corresponding files from the website and open them with "Chaos for Java"

- Recall the bifurcation or final state diagram of the logistic map from the last lecture.
- It is apparent that there is an infinite cascade of orbits with periods $1 \to 2 \to 2^2 \to 2^3 \to \cdots$, ending in chaos. It is known as the *period doubling route to chaos*, which we now start to examine in detail.

The first period doubling

- In Lecture 5 we looked at the first period doubling of the logistic map. For $r > 3$ there are two new fixed points of the second composition map f_2,

$$x_\pm^* = \frac{1 + r \pm \sqrt{r^2 - 2r - 3}}{2r}.$$

- We also saw that $f_2'(x_\pm^*) = 4 + 2r - r^2$; $f_2'(x_\pm) = 1$ when $r = 3$, and $f_2'(x_\pm) = -1$ when $r = 1 + \sqrt{6} \approx 3.4494897$. Hence $x_\pm^*$ is stable for $3 < r < 1 + \sqrt{6}$.
- At $r = 3$, the new fixed points coincide with the continuing fixed point x_1^* of f. As for derivatives,

$$f'(x_1^*) = -1, \qquad f_2'(x_1^*) = \left(f'(x_1^*)\right)^2 = +1.$$

  This is the reason for period doubling: When a fixed point loses stability through derivative value -1, the second composition map undergoes structural changes through derivative value $+1 = (-1)^2$.

Graphical point of view

- Consider the superposed graphs of f and f_2 for values of r a little below, and a little above, the critical value $r^* = 3$. (Overhead_10_1 & 10_2)
- The mechanism is very clear. So long as $f_2'(x_1^*) < 1$, the graph only intersects the line $y = x$ at one point in the vicinity of x_1^*. But when $f_2'(x_1^*) > 1$, it intersects at three nearby points.

- By simple graphical reasoning we expect that $f_2'(x_\pm^*) < 1$ at the two new fixed points, at least while r is close to r^*. This produces a stable period 2 orbit, since the new fixed points of f_2 are not fixed points of f.

Properties of derivatives

- The essential feature seen in the overheads is that $\phi_2(x)$ is approximated by a cubic polynomial near x^*. This is equivalent to the coincidence that, when $r = r^*$,

$$\phi_2(x_1^*) = 0, \qquad \phi_2'(x_1^*) = 0, \qquad \phi_2''(x_1^*) = 0.$$

What is surprising is that just two conditions for the function ϕ leads to three condtions for the period-doubled function ϕ_2, the new feature being a condition of equal curvature.

- The first two conditions we already know. Let's calculate the next derivative of ϕ_2, both as general formulae and evaluated at the fixed point itself. First, the general formula

$$\begin{aligned} f_2''(x) &= \frac{d}{dx}\left[f'(f(x)) \cdot f'(x)\right] \\ &= f''(f(x)) \cdot f'(x)^2 + f'(f(x)) \cdot f''(x), \end{aligned}$$

which gives, using the fact that $f(x^*) = x^*$ and $f'(x^*) = -1$,

$$\phi_2''(x^*) = -f_2''(x^*) = -f''(x^*)[f'(x^*)^2 + f'(x^*)] = 0.$$

This depends only on the fact that loss of stability arises from the passage of $f'(x^*)$ through the value -1.

Unfolding the bifurcation

- At the critical point, ϕ_2 is approximed by the cubic $\phi_2(x) \approx C(x - x^*)^3$. In order to approximate the behaviour near the critical value r^*, write $r = r^* + \delta r$, where $|\delta r| \ll 1$.
- One subtlety which should be noted is that I am going to expand about the fixed point $x^*(r)$, itself a function of r, determined by the condition $\phi(x^*) = 0$.
- The first two derivatives will no longer be zero when $r \neq r^*$. Introduce the lowest-order approximations

$$f'(x^*) \approx -1 - A\delta r, \tag{10.1}$$

and

$$\frac{\phi_2''(x^*)}{2} \approx B\delta r, \qquad \frac{\phi_2'''(x^*)}{6} \approx C.$$

The approximation (10.1) for f' implies the corresponding approximation for f_2':

$$\phi_2'(x^*) = 1 - f'(x^*)^2 \approx -2A\delta r.$$

- Substituting into the Taylor expansion of ϕ_2, the fixed points are approximated by the solutions of

$$\phi_2(x) \approx -2A\delta r(x - x^*) + B\delta r(x - x^*)^2 + C(x - x^*)^3 = 0, \qquad (10.2)$$

of which one is $x = x^*$. For the other pair

$$x_\pm^* \approx x^* \pm \sqrt{(2A/C)\delta r}. \qquad (10.3)$$

This is a sideways parabola, seen quite clearly in final state diagrams which were constructed earlier.

Two kinds of bifurcation

- We see from (10.3) that everything hinges on the relative signs of A and C, which represents the behaviour of $\phi_2'(x^*)$ and $\phi_2'''(x^*)$ near to $r = r^*$. In the case that $f'(x^*)$ is a decreasing function of r, $A > 0$. We cannot say whether C is positive or negative without specifying more about the map f.
- Let's put this question aside temporarily, after we note that the parabola may extend either to the left or right, depending on this. In addition, the stability properties will depend on these coefficients.

A stable period doubled orbit

- Positive C corresponds to a negative third derivative $f_2'''(x^*)$; positive A corresponds to loss of stability of x^* at the bifurcation. In this case the new pair $x_\pm^*$ appears for $\delta r > 0$.
- This is normal period doubling. We first check that the pair $x_\pm^*$ do form a period 2 orbit; using the simplest approximation for f, and substituting for $x_\pm^*$ from (10.3),

$$f(x_\pm^*) \approx f(x_\mp^*).$$

- For stability, we must calculate the derivative $f_2' = 1 - \phi_2'$ at $x^*_\pm$. ϕ_2 is approximated by the cubic (10.2), which we write in the factored form

$$\phi_2(x) \approx C(x - x^*)(x - x^*_-)(x - x^*_+).$$

This is easy to differentiate using the product rule,

$$\phi_2'(x) \approx C[(x - x^*_-)(x - x^*_+) + (x - x^*)(x - x^*_+) + (x - x^*)(x - x^*_-)],$$

after which we may substitute each of the fixed points in turn. In doing so, remember that we already found in (10.3) that

$$x^*_+ - x^* = x^* - x^*_- \approx \sqrt{(2A/C)\delta r}$$

which gives

$$f_2'(x^*) = 1 + 2A\delta r, \qquad f_2'(x^*_\pm) = 1 - 4A\delta r.$$

- There are two important conclusions to be drawn, remembering that A tells us the rate at which $f'(x^*)$ decreases through the critical value -1, as a function of r.

(i) The period doubled orbit $x^*_\pm$ is indeed stable with derivative $f_2'(x^*_\pm)$ which decreases from an initial value $+1$ at $r = r^*$.

(ii) The rate at which $f_2'(x^*_\pm)$ is initially decreasing is four times the rate at which $f'(x^*)$ was finally decreasing: therefore the next period doubling should happen much faster than the previous one.

Death of an unstable period doubled orbit

- Negative C corresponds to a positive third derivative $f_2'''(x^*)$. In this case the real pair $x^*_\pm$ of solutions of equation (10.3) exists for $\delta r < 0$ and disappear at $r = r^*$.
- But the stability analysis is unchanged, meaning that the period 2 orbit is unstable.
- This is a kind of *reverse bifurcation*, in which a stable fixed point becomes unstable because it is joined by an unstable periodic orbit.
- Most of this behaviour will go unobserved in a simple numerical experiment, except for the mysterious disappearance of a stable orbit.

Period doubling cascades

- We may replace f by any composition f_n, and f_2 by the corresponding doubled composition f_{2n}, at each step of the argument.
- Let's apply this first to the cascade which emanates from the fixed point x_1^*. It becomes unstable and period doubles at some critical parameter value, provided that the third derivative f_2''' has the correct sign. At the birth of this new orbit, $f_2'(x_\pm^*) = 1$; immediately afterward it has decreased below this value, losing stability when it passes through the critical value -1.
- Setting f_2 into the place previously occupied by f, its second composition f_4 will take the place previously occupied by f_2. Since $f_2'(x_\pm^*) = -1$ for either point on the period 2 orbit, $f_4'(x_\pm^*) = +1$ and $f_4''(x_\pm^*) = 0$.
- Provided that the third derivative value $f_4'''(x_\pm^*)$ is negative, each of the two fixed points of f_2 undergoes a period doubling bifurcation, to produce four new stable fixed points $(x_0^*, \cdots, x_3^*)$ of f_4. They are not fixed points of either f or f_2, hence they form a stable period 4 orbit of f.
- The bootstrap process continues, in an infinite cascade.

The full monty

- Suppose that we commence with a stable period-n orbit, where n is an odd number. Its stability is determined by $f_n'(x_i^*)$.
- As the parameter value is increased, this may also pass through the critical value -1; if it does so, and again assuming something about the value of the third derivative $f_{2n}'''(x_i^*)$, the composition map f_n period doubles to give a stable period $2n$ orbit.
- Once started, this cascade will generally continue to chaos.
- In fact, it is the most common mechanism for the destruction of stable periodic orbits of any period.
- A period 3 case is shown in Overhead_10_3 & 10_4.