Lecture 1 — Simplicity and complexity

Source material: Chapter 1

To reproduce overheads shown in lectures, download the corresponding files from the website and open them with "Chaos for Java"

Dynamical models

- Our understanding of the world has dramatically changed since the mathematical study of dynamical systems was inaugurated through the work of Kepler and Newton.
- In the physical sciences we can predict the motion of a space craft to within a few kilometers after several years. This comes through the use of a *dynamical model.* This is a conceptualisation — the state is described by functions which change in time.
- A *deterministic* dynamical model is one whose future states are uniquely determined from its present state by prescribed laws of evolution.
- The process of modelling takes into account only the important facts. For example, a typical model of the solar system considers the sun and planets as point masses moving under the sole effect of gravitational forces.

Population models

- Dynamical models have been used to study populations for more than a century.
- One of the simplest systems is a seasonally breeding organism whose generations do not overlap. We want to understand how the size x_{t+1} of a population in generation $t+1$ is related to the size x_t in the preceding generation.
- Often an adjustable parameter appears, accounting for, say, the net reproductive rate of the population.
- We may express such a scalar relationship in general form

  $$x_{t+1} = r(x_t, \lambda) \cdot x_t.$$

  The function $r(x)$ is the net reproductive rate.
- Suppose that $r(x)$ decreases from an initial value $r(0) = r$ to $r(x) = 0$ at a limiting

population number K. It is natural to let x measure the population as a fraction of K, so $0 \le x \le 1$ stands for $0 \le$ population $\le K$.

- A simple example is the logistic model, which uses a linear decrease of $r(x)$:

$$r(x) = r(1 - x), \qquad f(x) = rx(1 - x).$$

- Starting from some initial population x_0, this gives rise to the sequence of populations, at successive generations k,

$$x_{k+1} = rx_k(1 - x_k).$$

This is a deterministic dynamical model.

Examples of behaviour

- Using *Chaos for Java* one can examine the solution in a number of ways. I show the first 50 generations, commencing from an initial population $x_0 = 0.1$. Four different values of the parameter r are shown.
- For $r = 1.9$ the population rises rapidly to a steady value of about 0.47, a figure determined by crowding. (Overhead 1_1)
- For $r = 2.9$ the population again stabilises through a sequence of small boom and bust cycles which die out. (Overhead 1_2)
- For $r = 3.3$ the system stabilises on a permanent boom and bust cycle. (Overhead 1_3)
- For $r = 3.6$ the behaviour is extremely complex — in fact chaotic. (Overhead 1_4)
- Later, I shall explain what " chaotic" means; for now simply observe that iterations, in a histogram, appear to be randomly distributed. (Overhead 1_5)
- Imagine the implications for population control policies if such a simple model can generate such disparate outcomes, depending only on the policy settings!

Financial models

- Two professors of economics introduced a paper with the words "Imagine a bargaining model . . . in which each party has been instructed by higher headquarters to respond to each new offer by her opposite number with a counter-offer that is to be calculated from a simple reaction function . . . If the perfectly deterministic sequence of offers and counter-offers that must emerge from these simple rules were to begin to oscillate wildly and apparently at random, the negotiations could easily break down as each party . . . came to suspect the other side of duplicity and sabotage. Yet all that may be involved, as we will see, is the phenomenon referred to as chaos . . . ”

- In 1838, Thomas Tooke wrote that “the money market turns out always to be in unstable equilibrium”, an assertion which has been described as an “absurdity” by modern writers.
- Benoit Mandelbrot, who coined the word *fractal*, first observed the phenomenon of *scaling* in price changes and income distributions. Fractals will be treated briefly in later lectures; self-similarity and scaling is one of their hallmarks.

Lorenz: the end of weather prediction?

- Unpredictable and chaotic behaviour in deterministic systems were not widely appreciated until the advent of electronic computation.
- Lorenz considered the equations

$$\begin{aligned} \frac{dx}{dt} &= \sigma(y - x), \\ \frac{dy}{dt} &= rx - y - xz, \\ \frac{dz}{dt} &= xy - bz. \end{aligned} \tag{1.1}$$

- x, y and z are the state variables, σ, b and r are parameters which control the types of behaviour
- There are only two non-linear terms — xz and xy — without them all possible behaviour patterns would be simple to understand.

Strange attractors

- Lorenz found solutions which are *nonperiodic* — they never repeat.
- These solutions are also *sensitively dependent* on initial conditions — for all practical purposes, prediction of the state of the system is very limited.

- Furthermore, in the regime where chaotic solutions exist, then regardless of the initial conditions, they are all attracted to a *strange attractor*.
- Two typical solutions are shown (Overheads 1_6 & 1_7), numerically generated by *Chaos for Java*, with Lorenz' choice for the parameters,

$$b = 10, \qquad \sigma = 8/3, \qquad r = 28.$$

The orbits have become one of the icons of chaos — the *Lorenz butterfly*.

The butterfly effect

- Lorenz noticed that when he attempted to recompute a given orbit, using the same program on the same computer, he got a different result from the original.
- An accuracy problem — after a while, the two solutions don't seem related any more.
- For example, the two orbits shown differ only by a change of initial values of x in the fourth significant place. We can watch them evolve — best done on the computer in real time (Overhead 1_8)!
- The relative error quickly becomes as large as the quantities themselves, although different solutions have similar qualitative behaviour over relatively short time intervals.
- A strange attractor supplies a recognisable structure for the solutions.
- This effect, *sensitive dependence* of a system to the most infinitesimal changes of initial state is known as the *butterfly effect*, after the title of a talk by Lorenz: "Predictability: Does the flap of a Butterfly's Wings in Brazil set off a Tornado in Texas?"
- If Lorenz' equations do not allow long time prediction, why should more complicated dynamical models of the atmosphere do any better?

Complex behaviour of simple systems

- This course is an elementary introduction to the theory of dynamical systems and chaos.
- We want to understand some of the phenomena which are common across diverse systems, and investigate the mechanisms which make them so.
- The approach will combine relatively simple mathematics with computer experiments using the program *Chaos for Java*, which has been developed specifically for this purpose.