Department of Mathematics
Mathematics Math3062
Non-linear dynamics and chaos
BD/2/2005

ASSIGNMENT 6

For assessment: due in Tuesday September 27th

Scrappy and poorly presented assignments will not gain full marks — be sure also to give adequate explanation and data for all answers.

1. Consider the "Cubic #3 map",

$$f(x) = x(1 - p + px^2).$$

(i) Calculate the Schwarzian derivative $S[f]$, and show that $S[f] < 0$ on the entire interval $[0, 1]$ when $p > 1$.
(ii) The map has a symmetrical period-2 orbit for $p > 2$, whose two points are

$$x^*_\pm = \pm\sqrt{1 - 2/p}.$$

Show that it is created by period doubling of the fixed point $x^* = 0$, at $p = 2$.
(iii) Derive a *formula* for $f_2'(x^*_\pm)$, and prove that this orbit undergoes a bifurcation at $p = 3$ that is *not* period doubling.

2. (You do not need to attach printed graphs in answering this question. However, be careful to present all relevant data in an understandable form.)
Consider the "Cubic #2" map

$$f(x) = rx(1 - x^2)/\sqrt{3},$$

with $r = 4.5$ (the maximum).
(i) Use the "Graphical Analysis" window to locate the unstable period 2 and period 3 orbits of the map. From the numerical data (for the derivatives of f_2 and f_3 at the fixed points), calculate the Lyapunov exponents of these orbits using the *formula* for Lyapunov exponents.
(ii) Use the "Iterate(1d)" window of *Chaos for Java* to estimate numerical values of $L(x_0)$, using a sample size of 10^6, and a discard size of at least 10^3, for *four* different choices of x_0 chosen from the fixed points found numerically in (i). Are these results consistent with the hypothesis that $L(x_0)$ has a value independent of x_0? Why are they not consistent with the results of (i)?
(iii) Repeat (i) and (ii) for the logistic map with $r = 4$ (the maximum). Any surprises?

3. (Attach relevant printed graphs as part of your answer to this question — do *not* substitute hand-drawn sketches.)
The "Cubic #1" map is in a period 3 window at $p = 3.66$.
(i) Using the "Graphical Analysis" window demonstrate that there is a tangent bifurcation just before this p value.
(ii) Locate the critical p value to four decimal places using the "Fourier Analysis" window. Discard a large number of iterations before taking the samples so as to allow for convergence to any periodic orbit.

4. Consider the "Cubic #1" map

$$f(x) = 27rx^2(1-x)/16, \qquad (0 \le r \le 4)$$

(i) By examining the Lyapunov exponents using *Chaos for Java*, locate the values $\bar{r}_0, \cdots, \bar{r}_4$ for the superstable orbits of period $2^0, \cdots, 2^4$, to three decimal places. For example, one sees that $\bar{r}_2$ (the position of the superstable period-4 orbit) ≈ 3.445.
(ii) By examining graphs of composition maps, refine your estimates of these values to 8 decimal places. For example, to get a more accurate value for $\bar{r}_2$, we examine one of the fixed points of f_4, and adjust the value of r to make the derivative at the fixed point very small. Since $f_4'(x^*)$ is positive when $r = 3.445331130$ and negative when $r = 3.445331135$, this gives $\bar{r}_2 \approx 3.44533113$ to 8 decimal places.
(iii) Use your values of $\bar{r}_n$ to estimate the Feigenbaum universal constant δ, and also the critical value r_∞ at which the period doubling cascade ends (as per the lecture notes).
(iv) The maximum point $x_{\max} = 2/3$ of $f(x)$ should be one of the points on any superstable orbit. Use the above values of $\bar{r}_n$ to accurately estimate the corresponding d_n values for $n = 1, \cdots, 4$ (recall that d_n is the distance between $x_{\max}$ and $f_{2^{n-1}}(x_{\max})$. For example, you should see that $d_1 = 0.1558\cdots$.
(v) Use your values of d_n to estimate the Feigenbaum constant α.