Department of Mathematics
Mathematics Math3062
Non-linear dynamics and chaos
BD/2/2005

ASSIGNMENT 5

Tutorial session August 29th — not for assessment

1. Consider the "Cubic #3" map

$$f(x) = x(1 - p + px^2), \qquad 0 \le r \le 4.$$

It has bifurcations at $p = 2$ and $p = 3$.
(i) Show that the first bifurcation is period doubling. Specifically, show that the loss of stability is because $f'(x^*)$ passes through the value -1.
(ii) Show that the second bifurcation is not period doubling. Specifically, show that $f_2'(x_\pm^*)$ passes through the value $+1$. (You will need to obtain a formula for $x_\pm^*$ in order to evaluate $f_2'(x_\pm^*)$.)
(iii) Check these findings using *Chaos for Java*.

2. Consider the "Cubic #1" map

$$f(x) = 27rx^2(1 - x)/16, \qquad (0 \le r \le 4)$$

(i) By examining the Lyapunov exponents using *Chaos for Java*, locate the values $\bar{r}_0, \cdots, \bar{r}_4$ for the superstable orbits of period $2^0, \cdots, 2^4$, to three decimal places. For example, one sees that $\bar{r}_2$ (the position of the superstable period-4 orbit) ≈ 3.445.
(ii) By examining graphs of composition maps, refine your estimates of these values to 8 decimal places. For example, to get a more accurate value for $\bar{r}_2$, we examine one of the fixed points of f_4, and adjust the value of r to make the derivative at the fixed point very small. Since $f_4'(x^*)$ is positive when $r = 3.445331130$ and negative when $r = 3.445331135$, this gives $\bar{r}_2 \approx 3.44533113$ to 8 decimal places.
(iii) Use your values of $\bar{r}_n$ to estimate the Feigenbaum universal constant δ, and also the critical value r_∞ at which the period doubling cascade ends (as per the lecture notes).
(iv) The maximum point $x_{\max} = 2/3$ of $f(x)$ should be one of the superstable orbits. Use the above values of $\bar{r}_n$ to accurately estimate the corresponding d_n values for $n = 1, \cdots, 4$ (recall that d_n is the distance between $x_{\max}$ and the closest member of the 2^n cycle). For example, you should see that $d_1 = 0.1558\cdots$.
(v) Use your values of d_n to estimate the Feigenbaum constant α.
(vi) Should we have expected the same values of δ and α as for the logistic map?