Department of Mathematics
Mathematics Math3062
Non-linear dynamics and chaos
BD/2/2005

ASSIGNMENT 3

For assessment: due in Tuesday August 23rd

Scrappy and poorly presented assignments will not gain full marks — be sure also to give adequate explanation and data for all answers.

1. Consider the "Sine map" $f(x) = q\sin(\pi x)$ of *Chaos for Java*. Choose $q = 0.97$.
(i) Using the "Graphical analysis" option, find all fixed points of f, $f_2 \cdots f_5$, and the derivative value for each. Tabulate your data.
(ii) Construct a table showing the *number* of periodic orbits of each period from 1 to 5, as on page 35 of the text, or lecture notes page 21.
(iii) Returning to the data of part (i), group the fixed points found there into orbits (a pair of points for a period 2 orbit, etc.), and classify the stability of each using the derivative data.

2. Iterations of the tent map may be classified by sequences $LRRRLR\ldots$ according as the x values fall to the left (L) or right (R) of the maximum of $f(x)$ at $x = 1/2$. This was the subject of a question on the last assignment, and this question follows on from there. Repeat the analysis for arbitrary $t < 1$ by finding the 3-cycles of the standard tent map with $t < 1$. Show that there is a critical value

$$t_3 = \frac{1+\sqrt{5}}{4}$$

below which there are no three cycles.
(This question is a little tricky — the point is to first *derive* formulae for the three points on each type of orbit, assuming they are of a given type, then apply the restriction that L-points are $\leq 1/2$, R-points $\geq 1/2$. This is an inequality involving a cubic polynomial.)

3. The Lyapunov exponent of the standard tent map is given by

$$\lambda(x) = \ln 2t,$$

which is a continuous function of t for $0 < t \leq 1$. A "non-linear" tent map is implemented in *Chaos for Java* as the "TENT#2 map", using the function

$$f(x) = \begin{cases} t\left(\frac{3}{2}x + x^2\right), & 0 \leq x \leq \frac{1}{2}, \\ t\left(\frac{3}{2}(1-x) + (1-x)^2\right), & \frac{1}{2} \leq x \leq 1. \end{cases}$$

(i) Using *Chaos for Java*, examine the Lyapunov exponent for this function, as a function of t; pay particular attention to the jump at $t = 1/2$.
(ii) Derive a formula for $\lambda(x)$ in the range $0 \le t < \frac{1}{2}$, and also for $\lambda(x)$ when $t = \frac{1}{2} + 0$, that is, $\lim_{t \to \frac{1}{2}+} \lambda(x)$.
(iii) Explain what you observe by numerical experimentation, give a formula for the jump, and check it against numerical experiment.

4. Consider the function

$$f(x) = 1 - ax^2.$$

(i) Show that it is a unimodal map of the interval $[-1, 1]$ for the parameter range $0 \le a \le 2$.
(ii) Find the fixed points of the map, and show that the map undergoes period-doubling at $a = 3/4$.
(iii) By using the formula for the second composition map, and its fixed points, show that the period-2 orbit becomes unstable at $a = 5/4$.