REPRESENTABLE FUNCTORS AND OPERATIONS ON RINGS

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Introduction

The main aim of this article is to describe the mechanics of certain types of operations on rings (e.g. λ-operations on special λ-rings or differentiation operators on rings with derivation). *En route* we meet the very useful notion of a representable functor from rings to rings. If B, R are rings, then the set $\mathrm{Hom}_{\mathscr{R}}(B, R)$ of ring homomorphisms does not, in general, have a ring structure (unlike, for instance, the case where G, H are abelian groups, in which case $\mathrm{Hom}_{\mathscr{G}}(G, H)$ is naturally an abelian group). However, we shall show in § 1 that, if B also has a 'co-ring' structure (in which case we call it a *biring*) then this induces a natural ring structure on $\mathrm{Hom}_{\mathscr{R}}(B, R)$. In this case, the functor $R \mapsto \mathrm{Hom}_{\mathscr{R}}(B, R)$ is said to be represented by the biring B.

In § 2, we demonstrate that this functor has a left adjoint, which we denote by $R \mapsto B \odot R$ to bring out the analogy with the abelian group case (where $H \mapsto G \otimes H$ is left adjoint to $H \mapsto \mathrm{Hom}_{\mathscr{G}}(G, H)$).

When, in § 4, we come to discuss natural operations on a certain class of rings, there are many constructions we may perform. Given a collection T of operations on a ring, we may give T a ring structure by addition and multiplication of the values of operations. We are also interested in the value of an operation on the sum and product of two elements in terms of its value on those elements. This may be given by imposing a co-ring structure on T in which, for example, the effect of an operation on the sum of two elements is determined by the 'co-sum' of that operation on the tensor product of the elements. Finally, given two operations on a ring, we may form the composite of one followed by the other. We insist that the identity operation is in T. All these requirements add up to the notion of a 'biring triple' in § 3.

In § 4 we discuss the class of rings on which the operations in T act and in §§ 5, 6 we concentrate on the particular examples afforded by special λ-rings and rings with derivation. There follow three appendices which contain material supplementary to that in the main text which would otherwise have interrupted the flow of the exposition.

A point needs to be made concerning the constructions and methods of proof given. Since this article only exists because the notions of category theory allow the idea of a binary operation on a ring $R \times R \to R$ to be dualized giving a co-operation $R \to R \otimes R$, to give honour where it is due, all proofs et cetera should be given in terms of commutative diagrams. However, a classical training in modern algebra makes it more easy to talk about elements and to refer to a binary operation taking a, $b \in R$ into $a.b \in R$. All the constructions and proofs given could be explained in terms of maps or elements. For reasons of lucidity, we sometimes give one, sometimes the other, and often both. We leave it in the safe hands of the reader to construct the alternative method where only one is given. In conclusion, we remark that describing everything in terms of maps will give appropriate generalizations of many notions in other categories besides the category of rings.

NOTATION. Throughout the article, the word 'ring' is to be read as an abbreviation for 'commutative ring with unit', and 'ring homomorphism' for 'unit-preserving ring homomorphism'.

The ring of integers will be denoted by $\mathbf{Z}$, and for any ring R the unique ring homomorphism $\mathbf{Z} \to R$ will be denoted by j.

We shall denote the category of rings and ring homomorphisms by $\mathscr{R}$, and the category of sets and functions by $\mathscr{E}ns$.

In order that the category $\mathscr{R}$ should have coproducts (given by tensor product), it is essential to include the null ring O in $\mathscr{R}$; e.g. $\mathbf{Z}_2 \otimes \mathbf{Z}_3 = O$.

The dual of a category $\mathscr{C}$ will be denoted by $\mathscr{C}^{\mathrm{op}}$. If X and Y are objects of $\mathscr{C}$, the set of morphisms from X to Y will be denoted by $\mathrm{Hom}_{\mathscr{C}}(X, Y)$.

A covariant functor will be called *left exact* if it preserves difference kernels and products and *right exact* if it preserves difference cokernels and coproducts (**3**). A contravariant functor will be called *left exact* if it takes coproducts to products and difference cokernels to difference kernels.

We regard a terminal object as a product indexed over the empty set, and an initial object as a coproduct indexed over the empty set. It follows, for example, that a contravariant left exact functor takes initial objects to terminal objects.

1. Representable functors

The notion of ring, or, indeed, of any Ω-algebra (**1**), can be defined in any category with products and a terminal object (**9**). The category $\mathscr{R}$ of rings and ring homomorphisms has an initial object $\mathbf{Z}$, and coproducts; if R and S are rings, their coproduct in $\mathscr{R}$ is the tensor product $R \otimes S$. It follows that $\mathscr{R}^{\mathrm{op}}$ is a satisfactory category in which to define an Ω-algebra.

DEFINITION 1.1. A *biring* B is a ring for which B^{op} is a ring in $\mathscr{R}^{\mathrm{op}}$.

Alternatively we could say that B is a biring if the affine scheme Spec(B) is a ring in the category of schemes (**2**).

More explicitly, a biring B is a ring together with ring homomorphisms

$$\begin{aligned} \alpha &: B \to B \otimes B \quad &&\text{(co-addition)},\\ \mu &: B \to B \otimes B \quad &&\text{(co-multiplication)},\\ o &: B \to \mathbf{Z} \quad &&\text{(co-zero)},\\ \nu &: B \to B \quad &&\text{(co-additive inverse)},\\ \iota &: B \to \mathbf{Z} \quad &&\text{(co-unit)}, \end{aligned}$$

which define the co-ring structure. That is to say, they satisfy the axioms given in Appendix A. The symbols α, μ, o, ν, ι will be used generically for co-addition, co-multiplication, co-zero, co-additive inverse, co-unit respectively.

If B_1 and B_2 are birings, a ring homomorphism

$$f\colon B_1 \to B_2$$

is a *biring homomorphism* if

$$f^{\mathrm{op}}\colon B_2^{\mathrm{op}} \to B_1^{\mathrm{op}}$$

is a homomorphism of rings in $\mathscr{R}^{\mathrm{op}}$. That is to say, f 'commutes' with the biring structure. The explicit definition of what this means is given in Appendix A.

In this way we obtain a category $\mathscr{B}$ of birings and biring homomorphisms. If B_1 and B_2 are birings, we can give $B_1 \otimes B_2$ a canonical biring structure by defining

$$\begin{aligned} o &= o_1 \otimes o_2, \quad \iota = \iota_1 \otimes \iota_2, \quad \nu = \nu_1 \otimes \nu_2,\\ \alpha &= (1_1 \otimes t \otimes 1_2)(\alpha_1 \otimes \alpha_2),\\ \mu &= (1_1 \otimes t \otimes 1_2)(\mu_1 \otimes \mu_2), \end{aligned}$$

where $t\colon B_1 \otimes B_2 \to B_2 \otimes B_1$ is the interchange homomorphism defined by

$$t(b_1 \otimes b_2) = b_2 \otimes b_1 \quad (b_1 \in B_1,\ b_2 \in B_2).$$

It may be verified that α, μ, o, ν, ι satisfy the axioms of Appendix A, and that the canonical maps of B_1 and B_2 into $B_1 \otimes B_2$ are biring homomorphisms. It follows that $B_1 \otimes B_2$ is the coproduct of B_1 and B_2 in the category $\mathscr{B}$.

By taking

$$\alpha = \mu = \text{canonical isomorphism}\colon \mathbf{Z} \to \mathbf{Z} \otimes \mathbf{Z}, \quad \text{and} \quad o = \nu = \iota = 1_{\mathbf{Z}},$$

we may give $\mathbf{Z}$ a biring structure, albeit a trivial one. If B is a biring, the unique ring homomorphism $j\colon \mathbf{Z} \to B$ is easily verified to be a biring homomorphism, so that $\mathbf{Z}$ is an initial object in $\mathscr{B}$.

The whole *raison d'être* of the definition of biring rests upon the fact that any left exact contravariant functor $F\colon \mathscr{R} \to \mathscr{E}ns$ lifts to a functor $\mathscr{B} \to \mathscr{R}$.

Because of the left exactness we have $F(\mathbf{Z}) = \{e\}$, a one element set, and $F(R_1 \otimes R_2) = F(R_1) \times F(R_2)$. If B is a biring, then the functions

$$\begin{aligned} F(\alpha)&\colon\ F(B) \times F(B) \to F(B),\\ F(\mu)&\colon\ F(B) \times F(B) \to F(B),\\ F(o)&\colon\ \{e\} \to F(B),\\ F(\nu)&\colon\ F(B) \to F(B),\\ F(\iota)&\colon\ \{e\} \to F(B) \end{aligned}$$

are the addition, multiplication, zero, additive inverse, unit, respectively, of a ring structure on $F(B)$.

More particularly, the functor $\mathrm{Hom}_{\mathscr{R}}\colon \mathscr{R}^{\mathrm{op}} \times \mathscr{R} \to \mathscr{E}ns$ extends in a natural way to a functor $\mathscr{B}^{\mathrm{op}} \times \mathscr{R} \to \mathscr{R}$. That is to say, if B is a biring and $\mathscr{R}$ a ring, then the co-ring structure of B induces a ring structure on the set $\mathrm{Hom}_{\mathscr{R}}(B, R)$.

We denote the functor $R \mapsto \mathrm{Hom}_{\mathscr{R}}(B, R)$ by B^*, and such a functor from $\mathscr{R}$ to $\mathscr{R}$ we call *representable.*

It is important to note that the sum and product of two elements f, $g \in \mathrm{Hom}_{\mathscr{R}}(B, R)$ is *not* necessarily the 'pointwise' sum and product of f and g.

If $R \otimes R \xrightarrow{m} R$ is the homomorphism defined by $m(r \otimes r') = rr'$, then $f + g$ is the homomorphism

$$B \xrightarrow{\alpha} B \otimes B \xrightarrow{f \otimes g} R \otimes R \xrightarrow{m} R,$$

and fg is the homomorphism

$$B \xrightarrow{\mu} B \otimes B \xrightarrow{f \otimes g} R \otimes R \xrightarrow{m} R.$$

In terms of elements, we have

$$\begin{aligned} (f+g)(b) &= \sum_i f(b_i')g(b_i''),\\ (fg)(b) &= \sum_i f(\bar{b}_j)g(\bar{\bar{b}}_j), \end{aligned}$$

for any $b \in B$, where

$$\alpha(b) = \sum_i b_i' \otimes b_i'', \quad \mu(b) = \sum_j \bar{b}_j \otimes \bar{\bar{b}}_j.$$

Take care not to confuse the identity map $1_B \in \mathrm{Hom}_{\mathscr{R}}(B, B)$ with the unit element $B \xrightarrow{\iota} \mathbf{Z} \xrightarrow{j} B$ or with the zero element $B \xrightarrow{o} \mathbf{Z} \xrightarrow{j} B$ of $\mathrm{Hom}_{\mathscr{R}}(B, B)$.

We give some examples of representable functors from $\mathscr{R}$ to $\mathscr{R}$.

EXAMPLE 1. The identity functor $I_{\mathscr{R}}\colon \mathscr{R} \to \mathscr{R}$ is represented by the biring $I = \mathbf{Z}[e]$ given by

$$\begin{aligned}
\alpha(e) &= e \otimes 1 + 1 \otimes e,\\
\mu(e) &= e \otimes e,\\
o(e) &= 0,\\
\nu(e) &= -e,\\
\iota(e) &= 1.
\end{aligned}$$

The natural isomorphism $\mathrm{Hom}_{\mathscr{R}}(I, R) \to R$ assigns to the homomorphism $f\colon I \to R$ the element $f(e) \in R$. Conversely, for every $r \in R$ we may define a homomorphism $\hat{r}\colon I \to R$ by the condition $\hat{r}(e) = r$.

EXAMPLE 2. The formal power series ring functor

$$R \mapsto R[[t]]$$

is represented by the biring $P = \mathbf{Z}[X_0, X_1, \ldots]$ given by

$$\begin{aligned}
\alpha(X_n) &= X_n \otimes 1 + 1 \otimes X_n,\\
\mu(X_n) &= \sum_{r+s=n} X_r \otimes X_s,\\
o(X_n) &= 0,\\
\nu(X_n) &= -X_n,\\
\iota(X_m) &= 0 \quad \text{if } m > 0,\ \iota(X_0) = 1.
\end{aligned}$$

The natural isomorphism $\mathrm{Hom}_{\mathscr{R}}(P, R) \to R[[t]]$ assigns to the homomorphism $f\colon P \to R$ the formal power series $\sum_{n \geqslant 0} f(X_n) t^n$. The identity map 1_P corresponds to the generic power series $\sum_{n \geqslant 0} X_n t^n \in P[[t]]$, and every power series over R is the image of this under an appropriate homomorphism from P to R.

We note that the polynomial ring functor

$$R \mapsto R[t]$$

is not representable. For, if it were representable by a biring B, the identity map 1_B would correspond to a 'generic polynomial' in $B[t]$. We have only to ask ourselves what the degree of a generic polynomial is to see that there can be no such thing.

If $\varphi\colon B_1 \to B_2$ is a biring homomorphism, it induces a natural map $\varphi^*\colon B_2{}^* \to B_1{}^*$. A Yoneda-type argument ((3) 112) gives

PROPOSITION 1.2. *Every natural map $\mu\colon B_2{}^* \to B_1{}^*$ is of the form φ^* for a unique biring homomorphism $\varphi\colon B_1 \to B_2$.*

Proof. We define $\varphi = \mu_{B_2}(1_{B_2}) \in \operatorname{Hom}_{\mathscr{R}}(B_1, B_2)$, and then prove that φ is a biring homomorphism.

2. The functor $\odot$

In this section a functor will be defined which plays a central role in what follows. The symbol $\odot$ has been chosen to show an analogy with $\otimes$. Using this functor, we shall prove in this section that the composite of two representable functors from $\mathscr{R}$ to $\mathscr{R}$ is representable, and that a functor from $\mathscr{R}$ to $\mathscr{R}$ is representable if and only if it has a left adjoint.

Let B be a biring and R a ring. Define $B \odot R$ to be the quotient of the free ring on the set of ordered pairs (b, r) ($b \in B$, $r \in R$) by the ideal generated by all expressions of the form

$$\begin{aligned}
&(b_1+b_2, r)-(b_1, r)-(b_2, r),\\
&(b_1b_2, r)-(b_1, r)(b_2, r),\\
&(1, r)-1,\\
&(b, r_1+r_2)-\sum_i (b_i', r_1)(b_i'', r_2),\\
&(b, r_1r_2)-\sum_j (\bar{b}_j, r_1)(\bar{\bar{b}}_j, r_2),\\
&(b, -r)-(\nu(b), r),\\
&(b, 1)-\iota(b),\\
&(b, 0)-o(b),
\end{aligned}$$

where b, b_1, $b_2 \in B$, r_1, r_2, $r \in R$, and

$$\alpha(b) = \sum_i b_i' \otimes b_i'',$$

$$\mu(b) = \sum_j \bar{b}_j \otimes \bar{\bar{b}}_j.$$

We denote the image of (b, r) in $B \odot R$ by $b \odot r$.

Let $\varphi\colon B \to B'$ be a biring homomorphism and $f\colon R \to R'$ be a ring homomorphism.

We define a ring homomorphism

$$\varphi \odot f\colon\ B \odot R \to B' \odot R'$$

by $(\varphi \odot f)(b \odot r) = \varphi(b) \odot f(r)$.

It is easily verified that in this way we obtain a functor

$$\odot\colon\ \mathscr{B}\times\mathscr{R}\to\mathscr{R}.$$

We denote the functor $R\mapsto B\odot R$ by B_*.

THEOREM 2.1. *B_* is left adjoint to B^*. More generally, there is an isomorphism*

$$\theta(B,S,R)\colon\ \mathrm{Hom}_{\mathscr{R}}(B\odot S,R)\to\mathrm{Hom}_{\mathscr{R}}(S,\mathrm{Hom}_{\mathscr{R}}(B,R)),$$

natural in the three variables B ($\in\mathscr{B}$) and R, S ($\in\mathscr{R}$).

Proof. Let $f\colon B\odot S\to R$ be a ring homomorphism. We define $\theta(f)\colon S\to\mathrm{Hom}_{\mathscr{R}}(B,R)$ by

$$\{\theta(f)(s)\}(b)=f(b\odot s)\quad(b\in B,\ s\in S).$$

Conversely, a ring homomorphism $g\colon S\to\mathrm{Hom}_{\mathscr{R}}(B,R)$ defines

$$\theta^{-1}(g)\colon\ B\odot S\to R$$

by

$$\theta^{-1}(g)(b\odot s)=\{g(s)\}(b).$$

COROLLARY 2.2. *Since B_* is a left adjoint, it is right exact, i.e. it preserves co-products and initial objects* ((**9**) 528–32).

We can write this as

$$B_*(R\dot{\otimes}S)=B_*(R)\otimes B_*(S),\quad B_*(\mathbf{Z})=\mathbf{Z}.$$

It follows that, if R has a biring structure, so has $B_*(R)$. Hence, we obtain

COROLLARY 2.3. *$\odot\colon\mathscr{B}\times\mathscr{R}\to\mathscr{R}$ extends to a functor $\mathscr{B}\times\mathscr{B}\to\mathscr{B}$.*

THEOREM 2.4. *Let B, C be birings. Then*

$$(B\odot C)^*=C^*B^*,\quad(B\odot C)_*=B_*C_*.$$

If $\theta\colon B\to B'$, $\varphi\colon C\to C'$ are biring homomorphisms, then

$$(\theta\odot\varphi)^*=\varphi^*\cdot\theta^*,\quad(\theta\odot\varphi)_*=\theta_*\cdot\varphi_*.$$

Proof. For a ring R we have

$$\begin{aligned}C^*B^*(R)&=\mathrm{Hom}_{\mathscr{R}}(C,\mathrm{Hom}_{\mathscr{R}}(B,R))\\&=\mathrm{Hom}_{\mathscr{R}}(B\odot C,R)=(B\odot C)^*(R).\end{aligned}$$

The fact that $B_*C_*=(B\odot C)_*$ now follows from the uniqueness of left adjoints, and the rest of the theorem is trivial.

We can restate part of Theorem 2.4 as

COROLLARY 2.5. *The composite of two representable functors is representable, and $\odot$ is an associative operation.*

THEOREM 2.6. *A functor $F\colon \mathscr{R} \to \mathscr{R}$ is representable if and only if it has a left adjoint.*

Proof. The condition is necessary by Theorem 2.1. To prove sufficiency, let G be the left adjoint of F. Then G preserves co-products and initial objects ((**9**) 528–32) and so, if B is a biring, so is $G(B)$. For any biring R we have

$$F(R) = \operatorname{Hom}_{\mathscr{R}}(I, F(R)) = \operatorname{Hom}_{\mathscr{R}}(G(I), R) = G(I)^*(R).$$

So F is represented by the biring $G(I)$.

3. Biring triples

For any biring B, we may define biring homomorphisms $B \to B \odot I$, $B \to I \odot B$ by $b \mapsto b \odot e$, $b \mapsto e \odot b$ respectively. These may be shown to be isomorphisms of birings by writing down the obvious inverse map in each case.

DEFINITION 3.1. A *bi-ring triple* (T, π, η) consists of a bi-ring T and homomorphisms of birings

$$\pi\colon\ T \odot T \to T,$$
$$\eta\colon\ I \to T,$$

such that the following diagrams commute:

(1)
$$\begin{tikzcd}
T \odot T \odot T \arrow[r, "\pi \odot 1_T"] \arrow[d, "1_T \odot \pi"'] & T \odot T \arrow[d, "\pi"] \\
T \odot T \arrow[r, "\pi"] & T
\end{tikzcd}$$

(2)
$$\begin{tikzcd}
I \odot T \arrow[r, "\eta \odot 1_T"] \arrow[dr, "\wr"'] & T \odot T \arrow[d, "\pi"] & T \odot I \arrow[l, "1_T \odot \eta"'] \arrow[dl, "\wr"] \\
 & T &
\end{tikzcd}$$

Note that (T_*, π_*, η_*) is a triple and (T^*, π^*, η^*) is a co-triple in the usual sense ((**4**) (**8**)). Such triples and co-triples will be called *representable*. For the definitions of a triple, see Appendix B.

We shall indulge in the usual abuse of notation and refer to the biring triple T rather than (T, π, η). If x, $y \in T$, it will be convenient, and suggestive, to denote $\pi(x \odot y)$ by $x \cdot y$ and refer to this binary operation on T as composition. We also denote $\eta(e) \in T$ by the symbol e_T or simply e when there is no possibility of confusion. We call e_T the identity element of T (not to be confused with the unit element $1 \in T$). In § 4, it will be seen that elements of T correspond to operations on a certain class of rings. The elements $x+y$, xy will correspond to the pointwise sum and product respectively of the operations corresponding to x and y. The element $x \cdot y$ will correspond to the composite operation y followed by x. (The order of composition is because operations will be written on the left.) Diagram (1) indicates that composition is associative and (2) shows that e is a two-sided identity under composition. In fact e will correspond to the identity operation whereas the unit, $1 \in T$, will correspond to the operation sending every element of a given ring into the unit element of that ring. It is to be emphasized that π and η are biring homomorphisms and this imposes extra conditions on composition and on the element e.

A map from the biring triple (T, π, η) to (T', π', η') is a biring homomorphism $\theta\colon T \to T'$ such that the following diagrams commute:

(3)
$$\begin{tikzcd} T \odot T \arrow[r, "\theta \odot \theta"] \arrow[d, "\pi"] & T' \odot T' \arrow[d, "\pi'"] \\ T \arrow[r, "\theta"] & T' \end{tikzcd}$$

(4)
$$\begin{tikzcd} T \arrow[rr, "\theta"] & & T' \\ & I \arrow[ul, "\eta"] \arrow[ur, "\eta'"'] & \end{tikzcd}$$

Equivalently θ is a biring homomorphism such that $\theta(x \cdot y) = \theta(x) \cdot \theta(y)$ and $\theta(e_T) = e_{T'}$.

This gives a category $\mathscr{T}$ of biring triples and maps of biring triples. It is clear that the canonical isomorphism $I \odot I \to I$ makes I into an initial object for $\mathscr{T}$.

More generally, using the biring I, we may define a functor Ψ from the category of semigroups to the category of biring triples.

Let G be a semigroup with identity element e and let the composite of $g_1, g_2 \in G$ be denoted by $g_1 \cdot g_2$. Define $\Psi(G)$ to be the free commutative ring with unit on the symbols $g \in G$. Then we have canonical ring homomorphisms $u_g\colon I \to \Psi(G)$ given by $u_g(e) = g$ and we see that $\Psi(G) \simeq \bigotimes_{g\in G} I$. The biring structure on I induces a biring structure on $\Psi(G)$ in which $\alpha(g) = g\otimes 1 + 1\otimes g$, et cetera. We make $\Psi(G)$ into a biring triple by defining $\eta\colon I \to \Psi(G)$ to be u_e and $\pi\colon \Psi(G)\odot\Psi(G) \to \Psi(G)$ to be the biring homomorphism induced by the composition $\pi(g_1 \odot g_2) = g_1\cdot g_2$. In terms of maps, π is defined by the commutativity of diagram (5) for all pairs of elements g_1, $g_2 \in G$.

$$\text{(5)}\qquad \begin{tikzcd} I\odot I \arrow[r, "u_{g_1}\odot u_{g_2}"] & \Psi(G)\odot\Psi(G) \arrow[d, "\pi"] \\ I \arrow[u, "\wr"] \arrow[r, "u_{g_1\cdot g_2}"] & \Psi(G) \end{tikzcd}$$

The most trivial example of this construction is given by $\Psi(\{e\}) = I$.

Conversely, we may define a functor A from the category of biring triples to the category of semigroups. This requires a little explanation. Recall from § 1 Example 1 that, if T is a biring, it is canonically isomorphic to $\mathrm{Hom}_{\mathscr{R}}(I, T)$. Consider the subset $\mathrm{Hom}_{\mathscr{B}}(I, T)$ of $\mathrm{Hom}_{\mathscr{R}}(I, T)$ which consists of biring homomorphisms from I to T. If $x \in T$, we denote by $\hat{x}\colon I \to T$ the ring homomorphism defined by $\hat{x}(e) = x$.

From the definition of a biring homomorphism, we have immediately

LEMMA 3.2. *$\hat{x} \in \mathrm{Hom}_{\mathscr{B}}(I, T)$ if and only if*

$$\begin{aligned} \alpha(x) &= x\otimes 1 + 1\otimes x,\\ \mu(x) &= x\otimes x,\\ o(x) &= 0,\\ \nu(x) &= -x,\\ \iota(x) &= 1. \end{aligned}$$

We shall call an element of T satisfying the conditions of Lemma 3.2 *super-primitive*. If x and y are super-primitive, it may be verified that $x\cdot y$ is super-primitive and so the set of super-primitive elements is a semigroup which we denote by $A(T)$. The identity element of $A(T)$ is e.

Alternatively, this may be developed in terms of maps as follows.

If φ, $\psi \in \mathrm{Hom}_{\mathscr{B}}(I, T)$, we define $\varphi\cdot\psi \in \mathrm{Hom}_{\mathscr{B}}(I, T)$ by

$$(I \xrightarrow{\varphi\cdot\psi} T) = (I \xrightarrow{\sim} I\odot I \xrightarrow{\varphi\odot\psi} T\odot T \xrightarrow{\pi} T).$$

It is easily verified that this gives an associative binary operation on $\mathrm{Hom}_{\mathscr{B}}(I, T)$ with η as a two-sided identity element. In this way, using Lemma 3.2, we obtain a functor $A\colon \mathscr{T} \to \mathscr{S}grp$ (where $\mathscr{S}grp$ is the category of semigroups) given by $T \mapsto \mathrm{Hom}_{\mathscr{B}}(I, T)$.

THEOREM 3.3. *$\Psi\colon \mathscr{S}grp \to \mathscr{T}$ is left adjoint to*

$$A\colon \mathscr{T} \to \mathscr{S}grp.$$

Proof. Let G be a semigroup and T a triple. Given a semigroup homomorphism $\varphi\colon G \to A(T)$, we obtain a biring triple map $\tilde{\varphi}\colon \Psi(G) \to T$ defined by

$$I \xrightarrow{u_g} \Psi(G) \xrightarrow{\tilde{\varphi}} T = I \xrightarrow{\hat{\varphi}(g)} T.$$

Conversely, we need to prove that, given a map $h\colon \Psi(G) \to T$ of biring triples, we obtain a unique semigroup homomorphism $\varphi\colon G \to A(T)$ such that $\hat{\varphi} = h$. Since the elements of $\Psi(G)$ are polynomials in the symbols $g \in G$, we must show that the only super-primitive elements of $\Psi(G)$ are the symbols $g \in G$ themselves. The only elements of $\Psi(G)$ satisfying $\alpha(x) = x \otimes 1 + 1 \otimes x$ are those of the form $x = \sum_g C_g g$ (where C_g are integers). If also $\mu(x) = x \otimes x$, then $\sum_g C_g g \otimes g = \sum_g \sum_{g'} C_g C_{g'} g \otimes g'$, and so $C_g C_{g'} = 0$ if $g \neq g'$ and $C_g^2 = C_g$. This implies that at most one, say C_{g_0}, can be non-zero and, if it is non-zero, $C_{g_0} = 1$. Since $\iota(x) = 1$, we have $x \neq 0$ and so $x = g_0$ for some $g_0 \in G$.

This proves that $A\Psi(G) = G$ and hence that Ψ is left adjoint to A.

4. Modules over biring triples

DEFINITION 4.1. A *module over a biring triple* (T, π, η) is a pair (R, κ) where R is a ring and $\kappa\colon T \odot R \to R$ is a ring homomorphism such that the following diagrams commute:

(1)
$$\begin{tikzcd}
T\odot T\odot R \arrow[r, "\pi\odot 1_R"] \arrow[d, "1_T\odot\kappa"'] & T\odot R \arrow[d, "\kappa"] \\
T\odot R \arrow[r, "\kappa"] & R
\end{tikzcd}$$

(2)
$$\begin{tikzcd}
I\odot R \arrow[r, "\eta\odot 1_R"] \arrow[dr, "\sim"'] & T\odot R \arrow[d, "\kappa"] \\
 & R
\end{tikzcd}$$

We shall abuse language and call R a T-ring. In the usual notation ((4) (6)) for triples and co-triples, a T-ring is a T^*-comodule or a T_*-module (see Appendix B). We call κ the structure map of (R, κ). If x, $y \in T$, we have already decided to write $x \cdot y$ for $\pi(x \odot y)$; similarly, if R is a T-ring and $r \in R$, we shall denote $\kappa(x \odot r)$ by $x \cdot r$.

We may describe a T-ring R in terms of the pairing $T \times R \to R$ given by $(x, r) \mapsto x \odot r$. We define an action of $T \otimes T$ on $R \otimes R$ by $(x \otimes y) \cdot (r \otimes s) = (x \cdot r) \otimes (y \cdot s)$ and recall that the multiplication

$$m\colon\ R \otimes R \to R$$

is given by $m(r \otimes s) = rs$. Then diagrams (1), (2) are equivalent to the relations

$$(x \cdot y) \cdot r = x \cdot (y \cdot r),$$
$$e \cdot r = r,$$

for all x, $y \in T$, $r \in R$.

The fact that $x \odot r \mapsto x \cdot r$ is a ring homomorphism is equivalent to the relations

$$\begin{aligned}
(x \pm y) \cdot r &= x \cdot r \pm y \cdot r, \\
(xy) \cdot r &= (x \cdot r)(y \cdot r), \\
1 \cdot r &= 1, \\
x \cdot (r + s) &= m(\alpha(x) \cdot (r \otimes s)), \\
x \cdot (rs) &= m(\mu(x) \cdot (r \otimes s)), \\
x \cdot (-r) &= \nu(x) \cdot r, \\
x \cdot 1 &= \iota(x), \\
x \cdot 0 &= o(x),
\end{aligned}$$

for all x, $y \in T$, r, $s \in R$.

We may consider the elements of T as operations on R. Then $x \cdot y$ is composition of operations y followed by x and e is the identity operation. Sum and product of operations are performed pointwise; $1 \in T$ sends every element of R into $1 \in R$. The rest of the above axioms imply that the action of T on the ring structure of R is determined by the co-ring structure of T.

If (R, κ) and (R', κ') are both T-rings, a ring homomorphism $f\colon R \to R'$ is said to be a *map of T-rings* if the diagram

(3)
$$\begin{tikzcd}
T \odot R \arrow[r, "1_T \odot f"] \arrow[d, "\kappa"] & T \odot R' \arrow[d, "\kappa'"] \\
R \arrow[r, "f"] & R'
\end{tikzcd}$$

commutes.

This is equivalent to saying that f commutes with T-action, i.e. $f(x\cdot r) = x\cdot f(r)$, $x \in T$, $r \in R$.

In this way, we obtain a category $\mathscr{R}^T$ of T-rings and maps of T-rings.

The next proposition shows that $\mathscr{R}^T$ shares some of the satisfactory properties of $\mathscr{R}$.

PROPOSITION 4.2. *$\mathscr{R}^T$ has an initial object and coproducts.*

Proof. The canonical isomorphism $T \odot \mathbf{Z} \simeq \mathbf{Z}$ of Corollary 2.2. makes $\mathbf{Z}$ into a T-ring in a unique way. If R is a T-ring, the canonical ring homomorphism $j\colon \mathbf{Z} \to R$ is easily seen to be a map of T-rings. Hence $\mathbf{Z}$ is initial in $\mathscr{R}^T$.

If (R,κ) and (R',κ') are T-rings, then $R\otimes R'$ is made into a T-ring via the structure map

$$T \odot (R\otimes R') \simeq (T \odot R)\otimes(T \odot R') \xrightarrow{\kappa\otimes\kappa'} R\otimes R'.$$

The canonical injections of R and R' into $R\otimes R'$ are maps of T-rings and so $R\otimes R'$ is the co-product of R and R' in $\mathscr{R}^T$.

Given T-rings R, R', we may define an action of $T\otimes T$ on $R\otimes R'$ by $(t_1\otimes t_2)\cdot(r\otimes r') = (t_1\cdot r')\otimes(t_2\cdot r')$. On using the axiom

$$x\cdot(r_1 r_2) = m(\mu(x)\cdot(r_1\otimes r_2)),$$

where $r_1 = r\otimes 1$, $r_2 = 1\otimes r'$, Proposition 4.2 implies that the canonical T-action on $R\otimes R'$ is given by the rather baroque-looking identity

$$x\cdot(r\otimes r') = \mu(x)\cdot(r\otimes r').$$

We may make T itself into a T-ring by giving it the structure map $\pi\colon T \odot T \to T$. In Appendix C, it is shown that T may be thought of as the 'free' T-ring on one generator.

THEOREM 4.3. *If R is a T-ring, then there is a bijective map $\mathrm{Hom}_{\mathscr{R}^T}(T,R) \to R$ defined by $\varphi \mapsto \varphi(e)$, where $\varphi\colon T \to R$ is any map of T-rings.*

Proof. If $x \in T$ then $\varphi(x) = \varphi(x\cdot e) = x\cdot\varphi(e)$ and so φ is determined on the whole of T by its value on e and conversely.

COROLLARY 4.4. *Given a T-ring R and an element $r \in R$, there exists a unique map of T-rings $\varphi_r\colon T \to R$ in which $\varphi_r(e) = r$.*

Note the analogy between T considered as a T-ring and a ring A considered as a module over A. The element $e \in T$ occupies the analogous position to the unit $1 \in A$ (in that if M is an A-module, a module homomorphism $\theta\colon A \to M$ is defined uniquely by the element $\theta(1) \in M$).

We may consider the concept of a natural operation on the category $\mathscr{R}^T$. A natural operation σ assigns to each T-ring a map of *sets* $\sigma_R\colon R \to R$ such that, if $f\colon R \to R'$ is a map of T-rings, then the diagram

$$\begin{tikzcd} R \arrow[r, "\sigma_R"] \arrow[d, "f"'] & R \arrow[d, "f"] \\ R' \arrow[r, "\sigma_{R'}"] & R' \end{tikzcd} \tag{4}$$

commutes.

We denote the collection of natural operations by $\mathrm{Op}(\mathscr{R}^T)$. The elements of T may be considered as natural operations and it is a direct consequence of Corollary 4.4 that these are the only ones, for the natural operation σ corresponds to the element $\sigma_T(e) \in T$. We may induce a ring structure on the set of natural operations by addition and multiplication of values. Thus, if σ, τ are natural operations, then $\sigma+\tau$ and $\sigma\tau$ are defined by

$$(\sigma+\tau)_R(r) = \sigma_R(r)+\tau_R(r),$$
$$(\sigma\tau)_R(r) = \sigma_R(r)\tau_R(r),$$

for all $r \in R$. Of course there is another binary operation defined on $\mathrm{Op}(\mathscr{R}^T)$ given by composition of operations:

$$(\sigma\cdot\tau)_R(r) = \sigma_R(\tau_R(r)) \quad (r \in R).$$

Evidently we have

THEOREM 4.5. *$\mathrm{Op}(\mathscr{R}^T)$ is isomorphic to T as a ring, under the map $\sigma \mapsto \sigma_T(e)$. Composition of operations corresponds to composition in T and the identity operation corresponds to e.*

We shall now identify the elements of T with natural operations. As in Theorem 4.5, a natural operation σ corresponds to $\sigma_T(e) \in T$. Conversely, $x \in T$ corresponds to σ where $\sigma_R(r) = x\cdot r$ for any element r of the T-ring R. In terms of maps, x determines $\hat{x}\colon I \to T$ by $\hat{x}(e) = x$ and σ is determined by the identity

$$R \xrightarrow{\sigma_R} R = R \xrightarrow{\sim} I \odot R \xrightarrow{\hat{x}\odot 1_R} T \odot R \xrightarrow{\kappa} R$$

for a T-ring R.

Of particular interest are natural operations which are themselves ring homomorphisms. Such a natural operation we shall call an *Adams operation.* Since the composite of two ring homomorphisms is a ring homomorphism, the Adams operations form a semigroup under composition.

PROPOSITION 4.6. *The semigroup of Adams operations is*

$$A(T) = \mathrm{Hom}_{\mathscr{B}}(I, T).$$

Proof. The semigroup $A(T)$ is the subset of super-primitive elements. Now $x \in T$ is super-primitive if and only if $\hat{x}: I \to T$ is a biring homomorphism. In this case $\hat{x} \odot 1_R$ is a ring homomorphism and so, by the identity above, the corresponding operation σ_R is a ring homomorphism. Conversely, for any element $x \in T$, we have

$$\begin{aligned} x\cdot(e\otimes 1+1\otimes e) &= \alpha(x), \\ x\cdot(e\otimes e) &= \mu(x), \\ x\cdot(-e) &= \nu(x), \\ x\cdot 1 &= \iota(x), \\ x\cdot 0 &= o(x). \end{aligned}$$

If also $\sigma_R: R \to R$, given by $\sigma_R(r) = x\cdot r$, is a ring homomorphism for any ring R, we see that

$$\begin{aligned} x\cdot(e\otimes 1+1\otimes e) &= x\otimes 1+1\otimes x, \\ x\cdot(e\otimes e) &= x\otimes x, \\ x\cdot(-e) &= -x, \\ x\cdot 1 &= 1, \\ x\cdot 0 &= 0, \end{aligned}$$

and so x is super-primitive.

In § 3, we gave a functorial method of constructing a biring triple $\Psi(G)$ from a semigroup G. Using Proposition 4.6, we may describe the structure of a $\Psi(G)$-ring.

PROPOSITION 4.7. *Let G be a semigroup with identity, then a $\Psi(G)$-ring is a ring R together with a ring endomorphism $\theta_g: R \to R$ for each $g \in G$ such that $\theta_e = 1_R$ and $\theta_{g_1}\theta_{g_2} = \theta_{g_1\cdot g_2}$. A map of $\Psi(G)$-rings is a ring homomorphism $f: R \to R'$ such that $\theta'_g f = f\theta_g$ for each $g \in G$.*

Proof. $\Psi(G)$ is the polynomial ring in the symbols $g \in G$. Since $A\Psi(G) = G$, the super-primitive elements of $\Psi(G)$ are precisely the elements $g \in G$. Thus the natural operations corresponding to the elements $g \in G$ are ring homomorphisms and every other natural operation is a unique polynomial in these symbols. So all natural operations are determined by their action. Define $\theta_g: R \to R$ by $\theta_g(r) = g\cdot r$ and the proposition will follow.

As particular examples of this phenomenon, let $\mathbf{Z}^+$ be the additive semigroup of non-negative integers, $\mathbf{Z}$ the additive group of integers, and $\mathbf{Z}_2$ the group with two elements. Then a $\Psi(\mathbf{Z}^+)$-ring is a ring with endomorphism, a $\Psi(\mathbf{Z})$-ring is a ring with automorphism and a $\Psi(\mathbf{Z}_2)$-ring is a ring with an involutory automorphism.

5. Special λ-rings

We give a brief review of the theory of special λ-rings. The notion was first defined by Grothendieck (**7**). Examples of special λ-rings occur in K-theory (**5**). A full exposition may be found in (**6**).

A *λ-ring* is defined to be a commutative ring R with unit together with an enumerable family of maps (of the underlying set) $\lambda^n\colon R \to R$ $(n = 0, 1, 2, \ldots)$ satisfying, for all x, $y \in R$,

(1) $$\lambda^0(x) = 1,$$

(2) $$\lambda^1(x) = x,$$

(3) $$\lambda^n(x+y) = \sum_{r=0}^{n} \lambda^r(x)\lambda^{n-r}(y).$$

A *λ-homomorphism* is defined in the obvious way as a ring homomorphism commuting with the λ-operations.

If $1+R[[t]]^+$ denotes the set of formal power series in t, with coefficients in a ring R and constant term 1, then $1+R[[t]]^+$ has a unique λ-ring structure (**7**) subject to the following conditions.

(4) The structure is functorial in R.

(5) Addition in $1+R[[t]]^+$ is the usual multiplication of power series.

(6) Multiplication satisfies $(1+at)\circ(1+bt) = 1+abt$.

(7) $\lambda^n(1+at) = 1$ $(n \geqslant 2)$.

The 'zero' of $1+R[[t]]^+$ is 1 and the 'unit' is $1+t$. Alternatively, as described in ((**6**) Part I), this structure may be given in terms of 'universal polynomials'. If $\xi_1, \ldots, \xi_r, \eta_1, \ldots, \eta_s$ are indeterminates and σ_i, τ_i are the ith elementary symmetric functions in $\xi_1, \ldots, \xi_r$ and $\eta_1, \ldots, \eta_s$, respectively, let:

(8) $P_n(\sigma_1, \ldots, \sigma_n; \tau_1, \ldots, \tau_n)$ be the coefficient of t^n in
$$\prod_{i,j} (1+\xi_i\eta_j t);$$

(9) $P_{n,m}(\sigma_1, \ldots, \sigma_{nm})$ be the coefficient of t^n in
$$\prod_{i_1<\ldots<i_m} (1+\xi_{i_1}\ldots\xi_{i_m}t).$$

From (5), (6) and distributivity we have

(10) $$[\prod(1+\xi_i t)]\circ[\prod(1+\eta_j t)] = \prod(1+\xi_i\eta_j t),$$

which may be written as

(11) $$(1+\sum\sigma_n t^n)\circ(1+\sum\tau_n t^n) = 1+\sum P_n(\sigma_1, \ldots, \sigma_n; \tau_1, \ldots, \tau_n)t^n.$$

By naturality (4), a standard argument in symmetric function theory ((**6**) Part I) gives the following identity in $1+R[[t]]^+$.

$$(12)\qquad \Big(1+\sum_{n\geqslant 1} a_n t^n\Big)\circ\Big(1+\sum_{n\geqslant 1} b_n t^n\Big) = 1+\sum_{n\geqslant 1} P_n(a_1,\ldots,a_n;\, b_1,\ldots,b_n)t^n.$$

If x_1, ..., x_r are elements in an arbitrary λ-ring such that $\lambda^n(x_i)=0$ $(n\geqslant 2)$, then from (3) by induction on r,

$$(13)\qquad \lambda^m(\sum x_i) = \sum_{i_1<\ldots<i_m} x_{i_1}\ldots x_{i_m}.$$

Hence, using (5), (6), (7),

$$(14)\qquad \lambda^m(\prod(1+\xi_i t)) = \prod_{i_1<\ldots<i_m}(1+\xi_{i_1}\ldots\xi_{i_m}t),$$

which may be written as

$$(15)\qquad \lambda^m(1+\sum\sigma_n t^n) = 1+\sum P_{n,m}(\sigma_1,\ldots,\sigma_{nm})t^n.$$

Again, using naturality of the structure on $1+R[[t]]^+$,

$$(16)\qquad \lambda^m\Big(1+\sum_{n\geqslant 1} a_n t^n\Big) = 1+\sum_{n\geqslant 1} P_{n,m}(a_1,\ldots,a_{nm})t^n.$$

Note that if R is a λ-ring, the map $\lambda_t\colon R\to 1+R[[t]]^+$ given by $\lambda_t(x) = 1+\sum\lambda^n(x)t^n$ is homomorphic from the additive group of R to the 'additive' group of $1+R[[t]]^+$ by (1), (2), (3), and (5).

A λ-ring is said to be *special* if λ_t is a λ-homomorphism. Thus a λ-ring R is special if (1)–(3) are satisfied together with:

$$(17)\qquad \lambda_t(1) = 1+t,\quad\text{or}\quad \lambda^n(1)=0\quad (n\geqslant 2);$$

$$(18)\qquad \lambda^n(xy)\, P_n(\lambda^1(x),\ldots,\lambda^n(x);\, \lambda^1(y),\ldots,\lambda^n(y));$$

$$(19)\qquad \lambda^m(\lambda^n(x)) = P_{m,n}(\lambda^1(x),\ldots,\lambda^{mn}(x)).$$

Conditions (17)–(19) state that λ_t preserves the identity, preserves multiplication, and commutes with the λ-operations respectively.

THEOREM 5.1 (Grothendieck). *For any ring R, $1+R[[t]]^+$ is special.*

Proof. See (**6**).

The functor $R\mapsto 1+R[[t]]^+$ is representable. If $\Omega = \mathbf{Z}[s_1,\ldots,s_n,\ldots]$ is the free commutative ring on an enumerable set of generators, the co-ring structure is given by:

co-addition:

$$\alpha(s_n) = 1\otimes s_n + s_n\otimes 1 + \sum_{r=1}^{n-1} s_r\otimes s_{n-r},$$

co-multiplication:

$$\mu(s_n) = P_n(s_1 \otimes 1, \ldots, s_n \otimes 1;\ 1 \otimes s_1, \ldots, 1 \otimes s_n),$$

co-zero:

$$o(s_n) = 0,$$

co-unit:

$$\iota(s_1) = 1, \quad \iota(s_n) = 0 \quad (n > 1),$$

co-additive inverse:

$$\nu(s_1) = -s_1, \quad \text{and inductively} \quad \nu(s_n) = -s_n - \sum_{r=1}^{n-1} \nu(s_r) s_{n-r}.$$

The functor $R \mapsto 1 + R[[t]]^+$ is naturally equivalent to $R \mapsto \Omega^*(R)$ where $\varphi \in \Omega^*(R)$ is identified with $1 + \sum_{n \geq 1} \varphi(s_n) t^n \in 1 + R[[t]]^+$.

Define $\pi\colon \Omega \odot \Omega \to \Omega$ by

$$\pi(s_m \odot s_n) = s_m \cdot s_n = P_{m,n}(s_1, \ldots, s_{mn}), \tag{20}$$

and $\eta\colon I \to \Omega$ by

$$\eta(e) = s_1. \tag{21}$$

Since Theorem 5.1 is equivalent to the fact that $(\Omega^*, \pi^*, \eta^*)$ is a co-triple, it follows from Proposition 1.2 that (Ω, π, η) is a bi-ring triple.

Given an Ω-ring R, we may define a λ-ring structure on R by putting $\lambda^n(r) = s_n \cdot r$. From the definition of an Ω-ring, it follows immediately that the λ-structure on R satisfies (1)–(3) and (17)–(19) and so R is a special λ-ring. Conversely, a special λ-ring is an Ω-ring. Also maps of Ω-rings are precisely λ-homomorphisms.

From the results of § 4, we state Propositions 5.2–5.4 without proof.

PROPOSITION 5.2. *The tensor product of special λ-rings is a special λ-ring and the canonical maps $R_1 \to R_1 \otimes R_2$, $R_2 \to R_1 \otimes R_2$ are λ-homomorphisms.*

PROPOSITION 5.3. *The free special λ-ring on one generator is*

$$\Omega = \mathbf{Z}[s_1, \ldots, s_n, \ldots] \quad \textit{where } \lambda^n(s_1) = s_n.$$

If R is a special λ-ring, the λ-homomorphisms from Ω to R are in one-to-one correspondence with R under the relation $\theta \mapsto \theta(s_1)$.

A natural operation τ on the category of special λ-rings is a set map $\tau_R\colon R \to R$ for each special λ-ring R such that $f\tau_R = \tau_S f$ for each λ-homomorphism $f\colon R \to S$. The set of natural operations forms a ring under addition and multiplication of values.

PROPOSITION 5.4. *The ring of natural operations on the category of special λ-rings is isomorphic to Ω (as a ring) under the map $\tau \mapsto \tau(s_1)$.*

From this it follows that we may consider elements of Ω as natural operations. Composition of the operation $y \in \Omega$ followed by $x \in \Omega$ is given by $x \cdot y$ and the identity operation is $s_1 \in \Omega$. Since $s_n \cdot r = \lambda^n(r)$, the identification of Ω with the ring of natural operations amounts to identifying s_n and λ^n. An arbitrary natural operation τ takes $s_1 \in \Omega$ into another element of Ω, which is a polynomial with integer coefficients $\tau\Omega(s_1) = f(s_1, \ldots, s_n)$. This implies that $\tau = f(\lambda^1, \ldots, \lambda^n)$ with the above identification. Thus we have

COROLLARY 5.5. *A natural operation is a polynomial in the λ-operations.*

In the remainder of this section, we shall identify elements of Ω with natural operations and so an element of Ω is a polynomial in the λ-operations.

THEOREM 5.6. *$A(\Omega)$ is isomorphic to $\mathbf{Z}^\times$, the multiplicative semigroup of positive integers.*

Proof. We define a semigroup homomorphism

$$h\colon\ \mathbf{Z}^\times \to A(\Omega)$$

by

$$h(k) = \psi^k \quad (k > 0),$$

where ψ^k is the usual Adams operation (**5**) defined inductively by $\psi^1 = \lambda^1$ and

$$\psi^n = \sum_{r=1}^{n-1} (-1)^{r-1}\lambda^r\psi^{n-r} - (-1)^n n\lambda^n. \tag{22}$$

It is well known that the Adams operations are ring homomorphisms and that $\psi^m \cdot \psi^n = \psi^{mn}$ (**6**); so h is well defined.

Since the ψ^k are all distinct, h is a monomorphism.

Now h induces a map of biring triples

$$\tilde{h}\colon\ \Psi(\mathbf{Z}^\times) \to \Omega.$$

The essential point to notice is that, over the field of rationals $\mathbf{Q}$, the equation (22) can be solved to give the λ^n in terms of the ψ^k. Since the λ^n generate Ω, this means that

$$\tilde{h} \otimes 1_{\mathbf{Q}}\colon\ \Psi(\mathbf{Z}^\times) \otimes \mathbf{Q} \to \Omega \otimes \mathbf{Q}$$

is an isomorphism. It follows that, if a super-primitive element is not in the image of h, it must be a torsion element. But Ω has no torsion. Hence h is an epimorphism and the theorem is proved.

Alternatively, for the last part, we could use

PROPOSITION 5.7. *In any biring triple, a super-primitive element is not a torsion element.*

Proof. Let u be super-primitive and suppose that $nu = 0$ for some integer n. Then $n = n\iota(u) = \iota(nu) = 0$. Hence u is not a torsion element.

6. Rings with derivation

Let D be the biring $\mathbf{Z}[\delta_0, \ldots, \delta_n, \ldots]$ where

$$\alpha(\delta_n) = \delta_n \otimes 1 + 1 \otimes \delta_n,$$

$$\mu(\delta_n) = \sum_{q+s=n} \frac{n!}{q!\, s!} \delta_q \otimes \delta_s,$$

$$\nu(\delta_n) = -\delta_n,$$

$$o(\delta_n) = 0,$$

$$\iota(\delta_0) = 1, \quad \iota(\delta_n) = 0 \quad (n \geqslant 1).$$

Define $\eta: I \to D$ by $\eta(e) = \delta_0$ and $\pi: D \odot D \to D$ by $\pi(\delta_m \cdot \delta_n) = \delta_{m+n}$.

The functor $R \mapsto D^*(R)$ assigns to R the ring of formal power series of the type $\sum_{n \geqslant 0} a_n \frac{t^n}{n!}$ where $a_n \in R$. (Note that the fraction $1/n!$ is to be thought of as a normalization factor and we certainly do not expect to divide a_n by $n!$ in R.) The map $f: D \to R$ corresponds to the series $\sum f(\delta_n) \frac{t^n}{n!}$.

The homomorphism $\eta^*_R: D^*(R) \to R$ assigns to a power series its constant term and $\pi^*_R: D^*(R) \to D^*(D^*(R))$ is the homomorphism assigning to the power series $p(t)$ in one variable the power series $p(t_1 + t_2)$ in two variables t_1, t_2.

Suppose R is a D-ring. Then δ_0 is the identity operator, δ_1 is a derivation, and δ_n is the nth iterated power (under composition) of δ_1. Conversely, a ring with derivation can be given a canonical D-ring structure. For example, if S is the ring of real-valued C^∞ functions on the real line, then S has a natural D-ring structure given by $\delta_n \cdot f = f^{(n)}$.

A map of D-rings is precisely a ring homomorphism commuting with derivation.

The only Adams operation in D is δ_0.

Appendix A

The axioms for a ring B to be a biring are as follows.

Let $1: B \to B$ be the identity map of B, let $t: B \otimes B \to B \otimes B$ be the homomorphism defined by $t(x \otimes y) = y \otimes x$ $(x, y \in B)$, and let $m: B \otimes B \to B$ be the homomorphism defined by $m(x \otimes y) = xy$ $(x, y \in B)$. The following diagrams must be commutative.

B.1. Co-addition $\alpha\colon B \to B \otimes B$ is co-associative.

$$\begin{tikzcd}
B \arrow[r, "\alpha"] \arrow[d, "\alpha"'] & B \otimes B \arrow[d, "1 \otimes \alpha"] \\
B \otimes B \arrow[r, "\alpha \otimes 1"] & B \otimes B \otimes B
\end{tikzcd}$$

B.2. Co-addition is co-commutative.

$$\begin{tikzcd}
B \arrow[r, "\alpha"] \arrow[dr, "\alpha"'] & B \otimes B \arrow[d, "t"] \\
 & B \otimes B
\end{tikzcd}$$

B.3. Co-zero, $o\colon B \to \mathbf{Z}$.

$$\begin{tikzcd}
B \arrow[r, "\alpha"] \arrow[dd, "1"'] & B \otimes B \arrow[dr, "1 \otimes o"] & \\
 & & B \otimes \mathbf{Z} \arrow[dl, "1 \otimes j"] \\
B & B \otimes B \arrow[l, "m"'] &
\end{tikzcd}$$

B.4. Co-additive inverse, $\nu\colon B \to B$.

$$\begin{tikzcd}
B \arrow[r, "\alpha"] \arrow[d, "o"'] & B \otimes B \arrow[r, "1 \otimes \nu"] & B \otimes B \arrow[d, "m"] \\
\mathbf{Z} \arrow[rr, "j"] & & B
\end{tikzcd}$$

B.5. Co-multiplication $\mu\colon B \to B \otimes B$ is co-associative.

$$\begin{tikzcd}
B \arrow[r, "\mu"] \arrow[d, "\mu"'] & B \otimes B \arrow[d, "1 \otimes \mu"] \\
B \otimes B \arrow[r, "\mu \otimes 1"] & B \otimes B \otimes B
\end{tikzcd}$$

B.6. Co-multiplication is co-commutative.

$$\begin{tikzcd}
B \arrow[r, "\mu"] \arrow[dr, "\mu"'] & B \otimes B \arrow[d, "t"] \\
 & B \otimes B
\end{tikzcd}$$

B.7. Co-unit, $\iota \colon B \to \mathbf{Z}$.

$$\begin{tikzcd}
B \arrow[r, "\mu"] \arrow[dd, "1"'] & B \otimes B \arrow[dr, "1 \otimes \iota"] & \\
 & & B \otimes \mathbf{Z} \arrow[dl, "1 \otimes j"] \\
B & B \otimes B \arrow[l, "m"'] &
\end{tikzcd}$$

B.8. Co-distributivity.

$$\begin{tikzcd}
B \arrow[r, "\alpha"] \arrow[dd, "\mu"'] & B \otimes B \arrow[r, "\mu \otimes \mu"] & B \otimes B \otimes B \otimes B \arrow[d, "1 \otimes t \otimes 1"] \\
 & & B \otimes B \otimes B \otimes B \arrow[d, "1 \otimes 1 \otimes m"] \\
B \otimes B \arrow[rr, "\alpha \otimes 1"] & & B \otimes B \otimes B
\end{tikzcd}$$

If B_1 and B_2 are birings, and $f \colon B_1 \to B_2$ a ring homomorphism, then f is a biring homomorphism if the following diagrams commute.

$$\begin{tikzcd}
B_1 \arrow[r, "f"] \arrow[d, "\alpha_1"'] & B_2 \arrow[d, "\alpha_2"] \\
B_1 \otimes B_1 \arrow[r, "f \otimes f"] & B_2 \otimes B_2
\end{tikzcd}
\qquad
\begin{tikzcd}
B_1 \arrow[r, "f"] \arrow[d, "\mu_1"'] & B_2 \arrow[d, "\mu_2"] \\
B_1 \otimes B_1 \arrow[r, "f \otimes f"] & B_2 \otimes B_2
\end{tikzcd}$$

$$\begin{tikzcd}
B_1 \arrow[rr, "f"] \arrow[dr, "o_1"'] & & B_2 \arrow[dl, "o_2"] \\
 & \mathbf{Z} &
\end{tikzcd}
\qquad
\begin{tikzcd}
B_1 \arrow[rr, "f"] \arrow[dr, "\iota_1"'] & & B_2 \arrow[dl, "\iota_2"] \\
 & \mathbf{Z} &
\end{tikzcd}
\qquad
\begin{tikzcd}
B_1 \arrow[r, "f"] \arrow[d, "\nu_1"'] & B_2 \arrow[d, "\nu_2"] \\
B_1 \arrow[r, "f"] & B_2
\end{tikzcd}$$

Appendix B

A triple (F, p, q) on a category $\mathscr{C}$ is a functor $F: \mathscr{C} \to \mathscr{C}$ together with natural maps $p: F^2 \to F$, $q: I_{\mathscr{C}} \to F$ such that the following diagrams commute.

$$\begin{tikzcd} F^3 \arrow[r, "p*F"] \arrow[d, "F*p"'] & F^2 \arrow[d, "p"] \\ F^2 \arrow[r, "p"] & F \end{tikzcd} \qquad \begin{tikzcd} F \arrow[r, "F*q"] \arrow[dr, "\sim"'] & F^2 \arrow[d, "p"] & F \arrow[l, "q*F"'] \arrow[dl, "\sim"] \\ & F & \end{tikzcd}$$

A natural map $\theta: F \to F'$ is a map of triples if the diagrams

$$\begin{tikzcd} F^2 \arrow[r, "\theta^2"] \arrow[d, "p"'] & F'^2 \arrow[d, "p'"] \\ F \arrow[r, "\theta"] & F' \end{tikzcd} \qquad \begin{tikzcd} F \arrow[rr, "\theta"] & & F' \\ & I_{\mathscr{C}} \arrow[ul, "q"] \arrow[ur, "q'"'] & \end{tikzcd}$$

commute.

If (F, p, q) is a triple, an F-module is a pair (X, ξ) where X is an object of $\mathscr{C}$ and $\xi: F(X) \to X$ a map of $\mathscr{C}$ such that the diagrams

$$\begin{tikzcd} F^2(X) \arrow[r, "p_X"] \arrow[d, "F(\xi)"'] & F(X) \arrow[d, "\xi"] \\ F(X) \arrow[r, "\xi"] & X \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "q_X"] \arrow[dr, "\sim"'] & F(X) \arrow[d, "\xi"] \\ & X \end{tikzcd}$$

commute.

If (X, ξ) and (X', ξ') are F-modules, a map $f: X \to X'$ is a map of F-modules if the diagram

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\xi"'] & F(X') \arrow[d, "\xi'"] \\ X \arrow[r, "f"] & X' \end{tikzcd}$$

commutes.

If Y is an object of $\mathscr{C}$, then $(F(Y), p_Y)$ is easily verified to be an F-module, which we call the *free module* on Y.

Appendix C

Free modules of biring triples.

Let $\varphi\colon T \to U$ be a map of biring triples. We may define a 'pull-back' functor

$$\mathscr{R}^{\varphi}\colon \mathscr{R}^{U} \to \mathscr{R}^{T}$$

by $(R, \kappa) \mapsto (R, \kappa(\varphi \odot 1_R))$ for each U-ring (R, κ).

In particular, $\mathscr{R}^{\eta}\colon \mathscr{R}^{T} \to \mathscr{R}^{I} \simeq \mathscr{R}$ is simply the forgetful functor from T-rings to rings.

THEOREM. *$\mathscr{R}^{\varphi}$ has a right adjoint*

$$R \mapsto \operatorname{Hom}_{\mathscr{R}^T}(U, R)$$

and a left adjoint $R \mapsto U \odot_T R$, which will be defined below.

The notation has been chosen to suggest the obvious analogy! First of all, let us show that $\operatorname{Hom}_{\mathscr{R}^T}(U, R)$ is sensible notation.

Proposition 4.2 shows that $\mathscr{R}^{T}$ is a good category to define co-rings in. It also shows that $\mathscr{R}^{\varphi}$ is right exact. Hence, if X is a co-ring in $\mathscr{R}^{U}$, $\mathscr{R}^{\varphi}(X)$ is a co-ring in $\mathscr{R}^{T}$.

Now U is a co-ring in $\mathscr{R}^{U}$, and hence, by pull-back, a co-ring in $\mathscr{R}^{T}$ (we abuse notation by not distinguishing between $\mathscr{R}^{\varphi}(U)$ and U at this point). So the set $\operatorname{Hom}_{\mathscr{R}^T}(U, R)$ has a well-defined ring structure. We make it into a U-ring by defining

$$(u \cdot f)(u') = f(u \cdot u')$$

for all u, $u' \in U$, $f \in \operatorname{Hom}_{\mathscr{R}^T}(U, R)$.

Given a map of T-rings $\mathscr{R}^{\varphi}(S) \xrightarrow{h} R$, we define a map of U-rings $S \xrightarrow{g} \operatorname{Hom}_{\mathscr{R}^T}(U, R)$ by $\{g(s)\}(u) = h(u \cdot s)$. Conversely, given g we define h by $h(s) = \{g(s)\}(\eta(e))$.

If R is a T-ring, define $U \odot_T R$ to be the ideal generated by all expressions of the form

$$u \cdot \varphi(x) \odot r - u \odot x \cdot r$$

($u \in U$, $x \in T$, $r \in R$). Denote the image of $u \odot r$ in $U \odot_T R$ by $u \odot_T r$. We make $U \odot_T R$ into a U-ring by defining

$$u \cdot (u' \odot_T r) = u \cdot u' \odot_T r.$$

Given a map of T-rings $R \xrightarrow{h} \mathscr{R}^{\varphi}(S)$ we define a map of U-rings $U \odot_T R \xrightarrow{g} S$ by

$$g(u \odot_T r) = u \cdot h(r).$$

Conversely, given g we define h by

$$h(r) = g(\eta(e) \odot_T r).$$

In particular, the functor $T_*: \mathscr{R} \to \mathscr{R}^T: R \to T \odot R$ is left adjoint to the forgetful functor $\mathscr{R}^\eta: \mathscr{R}^T \to \mathscr{R}$, and so we call $T \odot R$ the *free* T-*ring* on the ring R.

The forgetful functor $\mathscr{R} \to \mathscr{E}ns$ has a left adjoint which sends $\{a, b, c, \ldots,\}$ to $\mathbf{Z}[a, b, c, \ldots]$. Since left adjoints preserve co-products and $S = \bigcup_{s \in S} \{s\}$ we see that the free ring on S is just

$$\bigotimes_{s \in S} \mathbf{Z}[s].$$

Hence, the free T-ring on a set S is

$$T \odot \left(\bigotimes_{s \in S} \mathbf{Z}[s]\right) \simeq \bigotimes_{s \in S} T,$$

since $T \odot \mathbf{Z}[s] \simeq T$.

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