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Arithmetic Analogues of Derivations

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In [1] the author introduced an arithmetic analogue of derivations, called π-derivations; this concept was used to prove a series of arithmetic analogues of results about algebraic differential equations [1, 2, 3]. Although the usefulness of this concept is probably well illustrated by these papers, the question arises of how "natural" these π-derivation operators are and what other choices one has in defining arithmetic analogues of "usual" derivations. The present note is an attempt to answer this question: we will introduce an *a priori* quite general notion of *jet operator* on a ring and prove that any such operator on a local domain of characteristic zero is "equivalent" to one of the following: a difference operator, a derivation operator, a π-*difference* operator (cf. the definition in the following text), or a π-derivation operator in the sense of [1]. The first three kinds of operators are always trivial on the rational integers; so one is left with the fourth kind, i.e., with the π-derivations of [1], as the "only" possibility, within our paradigm, for an arithmetic analogue of usual derivations. Throughout this paper rings are always assumed to be commutative, with unit element, and all ring homomorphisms preserve units.

DEFINITIONS. Let A be a ring. A set theoretic map $\delta: A \to A$ will be called an *operator* on A if there exist two polynomials $S, P \in A[X_0, X_1, Y_0, Y_1]$ such that for any $x, y \in A$ we have

$$\delta(x+y) = S(x, x\,\delta x, \delta y)$$
$$\delta(xy) = P(x, y, \delta x, \delta y).$$

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We say that δ and the pair (S, P) belong to each other. We say that δ is *generic* on A if, whenever $F \in A[X_0, X_1]$ is a polynomial such that

$$F(x, \delta x) = 0, \qquad x \in A,$$

we must have $F = 0$. By a *generic extension* of δ we understand an operator $\tilde{\delta}$ on a ring extension $\tilde{A}$ of A such that the following hold:

1. $\tilde{\delta}$ coincides with δ on A,
2. There exists a pair (S, P) belonging to both δ and $\tilde{\delta}$, and
3. $\tilde{\delta}$ is generic on $\tilde{A}$.

An operator δ on A will be called a *jet operator* if $\delta(0) = \delta(1) = 0$ and δ admits a generic extension. (The terminology will be justified by the Remark in the following text.)

Two operators δ_1, δ_2: $A \to A$ are said to be *equivalent* if there exist an invertible element $\lambda \in A^{\times}$ and a polynomial $f \in A[X]$ such that $f(0) = f(1) = 0$ and

$$\delta_1(x) = \lambda \cdot \delta_2(x) + f(x), \qquad x \in A.$$

If δ_1 and δ_2 are equivalent then δ_1 is a jet operator if and only if δ_2 is a jet operator.

EXAMPLES. There are four remarkable (series of) operators which we want to single out; under very mild assumptions on A these operators are actually jet operators (cf. the Remark in the following text). In what follows A will be any ring.

(a) Recall that a map δ: $A \to A$ is called a *difference operator* if it satisfies

$$\delta(x + y) = \delta x + \delta y$$
$$\delta(xy) = x\,\delta y + y\,\delta x + \delta x\,\delta y.$$

If A has no nontrivial idempotents then δ is a difference operator if and only if the map $x \mapsto x + \delta x$ is a ring homomorphism. Difference operators are the basis for "difference algebra" [5, 4]. (In these references difference operators are simply defined to be ring homomorphisms; for A without nontrivial idempotents, the two definitions lead, of course, to equivalent theories. We preferred a definition in which $\delta(1) = 0$.)

(b) Recall that a map δ: $A \to A$ is called a *derivation* if it satisfies

$$\delta(x + y) = \delta x + \delta y$$
$$\delta(xy) = x\,\delta y + y\,\delta x$$

Derivations are the basis for differential algebra [8, 6].

(c) Let $\pi \in A$ be noninvertible. A map $\delta\colon A \to A$ will be called a *π-difference operator* if it satisfies

$$\delta(x+y) = \delta x + \delta y$$
$$\delta(xy) = x\,\delta y + y\,\delta x + \pi\,\delta x\,\delta y.$$

If A has no nontrivial idempotents and π is a nonzero divisor then δ is a π-difference operator if and only if the map $x \mapsto x + \pi\,\delta x$ is a ring homomorphism.

(d) Let $\pi \in A$ be noninvertible, assume $\pi^* \in A$ is such that $\pi\pi^* = p$ where p is a prime integer, and let $q \neq 1$ be an integer power of p. Consider the polynomial with integral coefficients,

$$C_q(X,Y) = \left(X^q + Y^q - (X+Y)^q\right)/p.$$

A map $\delta\colon A \to A$ will be called a *π-derivation* [1] if it satisfies

$$\delta(x+y) = \delta x + \delta y + \pi^* C_q(x,y),$$
$$\delta(xy) = x^q\,\delta y + y^q\,\delta x + \pi\,\delta x\,\delta y.$$

If A has no nontrivial idempotents and π is a nonzero divisor then δ is a π-derivation if and only if the map $x \mapsto x^q + \pi\,\delta x$ is a ring homomorphism.

Remark. If A has no nontrivial idempotents and π is a nonzero divisor then all operators δ considered in the foregoing examples are jet operators. Indeed for any such δ one trivially checks that $\delta(0) = \delta(1) = 0$. On the other hand, if δ is of one of the four kinds of operators considered in the previous examples, then we may consider the ring,

$$\tilde{A} := A[T, T', T'', \dots, T^{(n)}, \dots],$$

of polynomials in infinitely many indeterminates and we may extend δ to an operator $\tilde{\delta}$ on $\tilde{A}$, of the same kind, such that

$$\delta T = T', \qquad \delta T' = T'', \dots, \delta T^{(n)} = T^{(n+1)}, \dots .$$

The possibility of the extension is well known and trivial to check in examples (a) and (b) (cf. Cohn [5] and Ritt and Kolchin [8, 6]), and it is also trivial to check in examples (c) and (d). Indeed, consider the ring homomorphism $\phi\colon A \to A$, $\phi(x) = x + \pi\,\delta x$ in case (c) and $\phi(x) = x^q + \pi\,\delta x$ in case (d), extend ϕ to a ring endomorphism $\tilde{\phi}$ of $\tilde{A}$ by letting

$$\tilde{\phi}(T) = T + \pi T',$$
$$\tilde{\phi}(T') = T' + \pi T'', \dots, \tilde{\phi}(T^{(n)}) = T^{(n)} + \pi T^{(n+1)}, \dots,$$

in case (c) and

$$\tilde{\phi}(T) = T^q + \pi T',$$
$$\tilde{\phi}(T') = (T')^q + \pi T'', \dots, \tilde{\phi}(T^{(n)}) = (T^{(n)})^q + \pi T^{(n+1)}, \dots,$$

in case (d), and finally define $\tilde{\delta}: \tilde{A} \to \tilde{A}$ by the formula $\tilde{\delta}(f) = (\tilde{\phi}(f) - f)/\pi$ in case (c) and $\tilde{\delta}(f) = (\tilde{\phi}(f) - f^q)/\pi$ in case (d). These formulae make sense since we assumed that π is a nonzero divisor in A. Now $\tilde{\delta}$ is generic on $\tilde{A}$. Indeed assume $F \in \tilde{A}[X_0, X_1]$ is such that $F(\tilde{x}, \delta\tilde{x}) = 0$ for all $\tilde{x} \in \tilde{A}$. Then let n be such that $F \in A[T, T', T'', \dots, T^{(n)}][X_0, X_1]$ and take $\tilde{x} = T^{(n+1)}$ to conclude that $F = 0$.

The ring $A[T, T', T'', \dots, T^{(n)}, \dots]$ should be viewed as the "ring of jets of the affine line"; this justifies our terminology.

It is worth noting that, in case $A = \mathbf{Z}, \mathbf{Z}_{(p)}, \mathbf{Z}_p$, $\pi = p$, of the four kinds of operators in the preceding text, p-derivations are the only nontrivial (i.e., nonidentically zero) ones.

Here is our main result.

THEOREM. *Assume A is a local integral domain of characteristic zero. Then any jet operator on A is equivalent to one of the following: a difference operator, a derivation operator, a π-difference operator, or a π-derivation operator.*

COROLLARY. *Any jet operator on $\mathbf{Z}_{(p)}$ (or on $\mathbf{Z}_p$) is equivalent to either $\delta = 0$ or to an operator δ of the form,*

$$\delta x = \frac{x - x^{p^m}}{p}.$$

The rest of the paper is devoted to the proof of the theorem. We start with a useful lemma.

LEMMA 1. *Let A be any ring. Assume $\delta: A \to A$ is an operator and $\delta: \tilde{A} \to \tilde{A}$ is a generic extension. Assume $F \in \tilde{A}[X_{1,0}, \dots, X_{m,0}, X_{1,1}, \dots, X_{m,1}]$ is a polynomial in $2m$ variables such that*

$$F\left(\tilde{x}_1, \dots, \tilde{x}_m, \tilde{\delta}\tilde{x}_1, \dots, \tilde{\delta}\tilde{x}_m\right) = 0,$$

for all $\tilde{x}_1, \dots, \tilde{x}_m \in \tilde{A}$. Then $F = 0$.

Proof. Trivial, by induction on m.

Note that the lemma immediately implies that, in its hypothesis, there exists exactly one pair (S, P) belonging to both δ and $\tilde{\delta}$.

LEMMA 2. *Let A be any ring. Assume $\delta\colon A \to A$ is a jet operator and $\delta\colon \tilde{A} \to \tilde{A}$ is a generic extension. Let (S, P) be the (unique) pair belonging to δ and $\tilde{\delta}$. Then for any A-algebra B, the formulae,*

$$(x_0, x_1) + (y_0, y_1) = (x_0 + y_0, S(x_0, y_0, x_1, y_1)), \tag{1}$$

$$(x_0, x_1) \cdot (y_0, y_1) = (x_0 y_0, P(x_0, y_0, x_1, y_1)), \tag{2}$$

define a ring structure on $B \times B$ such that the zero and unit elements in $B \times B$ are $(0,0)$ and $(1,0)$, respectively.

Proof. Let us check for instance that the addition on $B \times B$ is associative. We must check that the following two polynomials in 6 variables,

$$S(X_0 + Y_0, Z_0, S(X_0, Y_0, X_1, Y_1), Z_1) \quad \text{and}$$
$$S(X_0, Y_0 + Z_0, X_1, S(Y_0, Z_0, Y_1, Z_1))$$

are equal. By Lemma 1, it is sufficient to check that the previous polynomials become equal when $X_0, Y_0, Z_0, X_1, Y_1, Z_1$ are substituted by $\tilde{x}, \tilde{y}, \tilde{z}, \tilde{\delta}\tilde{x}, \tilde{\delta}\tilde{y}, \tilde{\delta}\tilde{z}$, for any $\tilde{x}, \tilde{y}, \tilde{z} \in \tilde{A}$. However, under this substitution, both polynomials become $\tilde{\delta}(\tilde{x} + \tilde{y} + \tilde{z})$.

The rest of the ring axioms, including the statement about the zero and the unit being $(0,0)$ and $(1,0)$, can be proved similarly. By the way, the inverse for addition is given by

$$-(x_0, x_1) = (-x_0, P(-1, x_0, \delta(-1), x_1)).$$

Remark. If in Lemma 2 we start with one of the examples (a), (b), (c), (d) then the ring structures on $B \times B$ one gets are familiar ones. In the case of example (a) the ring structure on $B \times B$ is isomorphic to the twofold product of B in the category of rings. In the case of example (b) the ring structure on $B \times B$ is, of course, isomorphic to the ring structure on the dual numbers $B[t]/(t^2)$. In the case of example (d) the ring structure on $B \times B$ is a version of what one calls "ramified Witt vectors." Case (c) leads to a similarly familiar ring.

Let A be any ring and let $\mathbf{A}^1_A = \operatorname{Spec} A[X_0]$, $\mathbf{A}^2_A = \operatorname{Spec} A[X_0, X_1]$ be the affine line and the affine plane over A. Denote by $\Sigma(A)$ the set of all ring A-scheme structures on $\mathbf{A}^2_A$ such that the first projection $pr_1\colon \mathbf{A}^2_A \to \mathbf{A}^1_A$ is a ring A-scheme homomorphism and such that the zero and the unit elements in $\mathbf{A}^2_A$ correspond to $(0,0)$, $(1,0)$. We can identify $\Sigma(A)$ with the set of all pairs (S, P) where $S, P \in A[X_0, X_1, Y_0, Y_1]$ such that the formulae (1) and (2) in Lemma 2 define, for any A-algebra B, a ring structure on $B \times B$ whose zero and unit are $(0,0)$ and $(1,0)$.

On the other hand, consider the group $\Gamma(A)$ of all A-scheme automorphisms σ of $\mathbf{A}^2_A$ that fix $(0,0)$ and $(1,0)$, and for which $pr_1 \circ \sigma = pr_1$. Assume in addition that A is an integral domain of characteristic zero. (This will be the case we shall consider in applications.) Then looking at Jacobian matrices, one immediately sees that any such σ is given by

$$(x_0, x_1) \mapsto (x_0, \lambda x_1 + f(x_0)),$$

for some $\lambda \in A^{\times}$ and some $f \in A[X]$ with $f(0) = f(1) = 0$. So we can identify $\Gamma(A)$ with $A^{\times} \times (X^2 - X)A[X]$, the latter equipped with the semidirect product group structure,

$$(\lambda_1, f_1) \cdot (\lambda_2, f_2) = (\lambda_1 \lambda_2, f_1 + \lambda_1 f_2).$$

Now $\Gamma(A)$ acts on $\Sigma(A)$ by transport of structure. Explicitly, if $(\lambda, f) \in \Gamma(A)$ and $(S, P) \in \Sigma(A)$ then we have

$$(\lambda, f) \cdot (S, P) = (S', P'),$$

where

$$S' = \lambda S\big(X_0, Y_0, \lambda^{-1}[X_1 - f(X_0)], \lambda^{-1}[Y_1 - f(Y_0)]\big) + f(X_0 + Y_0), \tag{3}$$

$$P' = \lambda P\big(X_0, Y_0, \lambda^{-1}[X_1 - f(X_0)], \lambda^{-1}[Y_1 - f(Y_0)]\big) + f(X_0 Y_0). \tag{4}$$

Lemma 2 shows that, if (S, P) is the pair belonging to a jet operator δ on A and to a generic extension of it, then $(S, P) \in \Sigma(A)$ and the map $A \to A \times A$, $x \mapsto (x, \delta x)$ is a ring homomorphism (where the ring structure on $A \times A$ is defined by (S, P) as in Lemma 2). Conversely, if we are given an operator ∂ on A such that $A \to A \times A$, $x \mapsto (x, \partial x)$ is a ring homomorphism (where the ring structure on $A \times A$ is defined by (S, P) as in Lemma 2) then ∂ belongs to (S, P).

So, in order to prove our theorem it is enough to prove the following

PROPOSITION. *Assume A is a local integral domain of characteristic zero. Then any element in $\Sigma(A)$ is $\Gamma(A)$-conjugate to a pair (S, P) belonging to the following list*

$$S = X_1 + Y_1, \qquad P = X_0 Y_1 + Y_0 X_1 + X_1 Y_1,$$
$$S = X_1 + Y_1, \qquad P = X_0 Y_1 + Y_0 X_1,$$
$$S = X_1 + Y_1, \qquad P = X_0 Y_1 + Y_0 X_1 + \pi X_1 Y_1,$$
$$S = X_1 + Y_1 + \pi^* C_q(X_0, Y_0), \qquad P = X_0^q Y_1 + Y_0^q X_1 + \pi X_1 Y_1.$$

The element π in the last two pairs in the preceding equations is as in the Examples (c) and (d), respectively.

The first step in the proof of the proposition is the following observation. Assume A is as in the proposition and let K be its field of fractions.

LEMMA 3. *Any element in $\Sigma(K)$ is $\Gamma(K)$-conjugate to one of the pairs,*

$$S = X_1 + Y_1, \qquad P = X_0Y_1 + Y_0X_1 + X_1Y_1,$$
$$S = X_1 + Y_1, \qquad P = X_0Y_1 + Y_0X_1.$$

Proof. Note that any element of $\Sigma(K)$ defines, in particular, a commutative algebraic group structure on $\mathbf{A}^2_K$ with zero element $(0,0)$ such that the first projection to $\mathbf{A}^1_K$ is a morphism of algebraic groups. Since the kernel of the first projection is isomorphic, as a variety, to $\mathbf{A}^1_K$, and since, by the theory of algebraic groups, any algebraic group structure on $\mathbf{A}^1_K$ is isomorphic to the usual additive group structure, it follows that our algebraic group $\mathbf{A}^2_K$ is an extension of the additive group by itself. By [9], p. 171, this extension is trivial. So any pair $(S, P) \in \Sigma(K)$ is conjugate, under the action of $\Gamma(K)$, to a pair (S', P'), where $S' = X_1 + Y_1$. By distributivity and commutativity in the ring structure axioms P' must be a bilinear symmetric form, i.e.,

$$P' = \alpha X_0Y_0 + \beta(X_0Y_1 + X_1Y_0) + \gamma X_1Y_1.$$

Because $P'(1, Y_0, 0, Y_1) = Y_1$ we get $\alpha = 0$ and $\beta = 1$. If $\gamma = 0$ then $(S', P') = (X_1 + Y_1, X_0Y_1 + Y_0X_1)$ and we are done. If $\gamma \neq 0$ note that

$$(S', P') = (\gamma^{-1}, \gamma^{-1}X)(X_1 + Y_1, X_0Y_1 + Y_0X_1 + X_1Y_1),$$

and we are done again.

Proof of the Proposition. By Lemma 3, there are two cases to examine for a pair $(S, P) \in \Sigma(A)$.

Case 1. (S, P) is $\Gamma(K)$-conjugate to $(X_1 + Y_1, X_0Y_1 + Y_0X_1)$.

In this case we claim that (S, P) is actually $\Gamma(A)$-conjugate to

$$(X_1 + Y_1, X_0Y_1 + Y_0X_1).$$

Indeed write

$$(S, P) = (\lambda, f)(X_1 + Y_1, X_0Y_1 + Y_0X_1),$$

where $\lambda \in K^\times$ and $f \in (X^2 - X)K[X]$. By the formulae (3), (4) we have

$$S = X_1 + Y_1 + f(X_0 + Y_0) - f(X_0) - f(Y_0),$$
$$P = X_0Y_1 + Y_0X_1 + f(X_0Y_0) - X_0f(Y_0) - Y_0f(X_0).$$

The preceding formulae show that we may assume $\lambda = 1$. Also, since P has coefficients in A, it follows immediately that f has coefficients in A. So $(\lambda, f) \in \Gamma(A)$ and our claim is proved.

Case 2. (S, P) is $\Gamma(K)$-conjugate to $(X_1 + Y_1, X_0Y_1 + Y_0X_1 + X_1Y_1)$. Write

$$(S, P) = (\lambda, f)(X_1 + Y_1, X_0Y_1 + Y_0X_1 + X_1Y_1),$$

where $\lambda \in K^\times$ and $f \in (X^2 - X)K[X]$. By the formulae (3), (4) we have

$$S = X_1 + Y_1 + f(X_0 + Y_0) - f(X_0) - f(Y_0), \tag{5}$$

$$\begin{aligned} P = X_0Y_1 + Y_0X_1 - X_0f(Y_0) - Y_0f(X_0) \\ + \lambda^{-1}[X_1Y_1 - X_1f(Y_0) - Y_1f(X_0) + f(X_0)f(Y_0)] + f(X_0Y_0). \end{aligned} \tag{6}$$

Setting $f^*(X) = f(X) - \lambda X$ we obtain

$$P = \lambda^{-1}[X_1Y_1 - X_1f^*(Y_0) - Y_1f^*(X_0) + f^*(X_0)f^*(Y_0)] + f^*(X_0Y_0). \tag{7}$$

Since P has coefficients in A, looking at the coefficients of X_1Y_1 and $X_1Y_0^i$ in f^* we get that $\lambda^{-1} \in A$ and $\lambda^{-1}f^* \in XA[X]$.

If $\lambda \in A$ we have $(\lambda, f) \in \Gamma(A)$ and we are done.

Hence we may assume from now on that $\lambda \notin A$, i.e., $\pi := \lambda^{-1} \in M$, where M is the maximal ideal of A. Let us write $f = f_1X + f_2X^2 + \cdots + f_mX^m$.

Assume for a moment that $f_i \in A$ for all $i \geq 2$. Since $f(1) = 0$ it follows that $f_1 \in A$ as well. In this case note that

$$(1, -f)(S, P) = (X_1 + Y_1, X_0Y_1 + Y_0X_1 + \pi X_1Y_1).$$

Since $(1, -f) \in \Gamma(A)$ we are done in this case.

So from now on we may assume that the following condition holds

$$\text{There exists an index } i_0 \geq 2 \text{ such that } f_{i_0} \notin A. \tag{8}$$

Recall that by a *cocycle* $G \in A[X, Y]$ one understands a polynomial G that satisfies

$$G(Y, Z) - G(X + Y, Z) + G(X, Y + Z) - G(X, Y) = 0.$$

By the *coboundary* associated to a polynomial $\phi \in A[X]$ one understands the polynomial in two variables,

$$(\partial\phi)(X,Y) := \phi(X+Y) - \phi(X) - \phi(Y).$$

By (6), since S has coefficients in A, it follows that

$$G(X_0,Y_0) := f(X_0+Y_0) - f(X_0) - f(Y_0)$$

is a cocycle in $A[X_0,Y_0]$. We claim that $M \cap \mathbf{Z} \neq 0$, for if the contrary holds then $\mathbf{Q}$ is contained in A so by [7] p. 257, Lemma 3, we must have $G = \partial\phi$ for some $\phi \in A[X]$. Hence $f - \phi$ would be additive, hence $f - \phi$ would be a monomial of degree one and this contradicts assumption (8). So we have $M \cap \mathbf{Z} = p\mathbf{Z}$ for some prime integer p. By [7] loc. cit. again, there exist a polynomial $\phi \in A[X]$ and there exist $a_0, a_1, a_2, \ldots \in A$ such that

$$G(X_0,Y_0) = \phi(X_0+Y_0) - \phi(X_0) - \phi(Y_0) + \sum_{j\geq 1} a_j p^{-1}\Big[(X_0+Y_0)^{p^j} - X_0^{p^j} - Y_0^{p^j}\Big].$$

Again it follows that the polynomial $f - \phi - \sum_{j\geq 1} a_j p^{-1} X^{p^j}$ is additive, hence it is a monomial of degree one. Hence we have $pf_i \in A$ for all $i \geq 2$. Let $\pi := \lambda^{-1}$. Since, by (7), the coefficient $\pi f_{i_0}^2 + f_{i_0}$ of $X_0^{i_0} Y_0^{i_0}$ in P is in A, (and since A is local), we must have $\pi f_{i_0} \in A^{\times}$ so we get $p \in \pi A$. Set $F = -\pi f^*$. As we have seen $F \in A[X]$ and note that the reduction of F modulo π, $\overline{F} \in (A/\pi A)[X]$, has degree ≥ 2 because the coefficient of X^{i_0} in F is invertible. Multiplying the equations (5) and (7) by π and reducing modulo π we find that $\overline{F}(X_0+Y_0) = \overline{F}(X_0) + \overline{F}(Y_0)$ and $\overline{F}(X_0Y_0) = \overline{F}(X_0)\overline{F}(Y_0)$. The latter (multiplicativity) relation implies (due to the fact that $A/\pi A$ is local) that $\overline{F} = X^m$ for some m. By further reducing modulo M and using the additivity relation for $\overline{F}$ we get that $m = q \neq 1$ is an integer power of p. Consequently, we may write $F = X^q + \pi g$, $g \in A[X]$. Then a straightforward computation shows that (S,P) equals

$$(1,-g)\Big(X_1 + Y_1 + \pi^{-1}\big[X_0^q + Y_0^q - (X_0+Y_0)^q\big], X_0^q Y_1 + Y_0^q X_1 + \pi X_1 Y_1\Big).$$

This completes the proof of the proposition and hence of the theorem.

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