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Plethystic algebra

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Abstract

The notion of a **Z**-algebra has a non-linear analogue, whose purpose it is to control operations on commutative rings rather than linear operations on abelian groups. These *plethories* can also be considered non-linear generalizations of cocommutative bialgebras. We establish a number of category-theoretic facts about plethories and their actions, including a Tannaka–Krein-style reconstruction theorem. We show that the classical ring of Witt vectors, with all its concomitant structure, can be understood in a formula-free way in terms of a plethystic version of an affine blow-up applied to the plethory generated by the Frobenius map. We also discuss the linear and infinitesimal structure of plethories and explain how this gives Bloch's Frobenius operator on the de Rham–Witt complex.
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Consider an example from arithmetic. Let p be a prime number. Recall that for (commutative) rings R, the ring $W(R)$ of (p-typical) Witt vectors is usually defined to be the unique ring structure on the set $R^{\mathbf{N}}$ which is functorial in R and such that the map

$$(r_0, r_1, \ldots) \mapsto (r_0, r_0^p + pr_1, r_0^{p^2} + pr_1^p + p^2 r_2, \ldots)$$

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is a ring homomorphism, the target having the usual, product ring structure. If R is a perfect field of characteristic p, then $W(R)$ is the unique complete discrete valuation ring whose maximal ideal is generated by p and whose residue field is R. However, in almost all other cases, $W(R)$ is pathological by the usual standards of commutative algebra. For example, $W(\mathbf{F}_p[x])$ is not noetherian.

It is nevertheless an established fact that $W(R)$ is an important object. For example, if R is the coordinate ring of a smooth affine variety over a perfect field of characteristic p, there is a certain quotient of the de Rham complex of $W(R)$, called the de Rham–Witt complex of R, whose cohomology is naturally the crystalline cohomology of R. But it is not at all clear from the definition above what the proper way to think about $W(R)$ is, much less why it is even reasonable to consider it in the first place. The presence of certain natural structure, for example, a multiplicative map $R \to W(R)$ and a ring map $W(R) \to W(W(R))$ adds to the mystery. And so we have a question: is there a definition given purely in terms of algebraic structure rather than somewhat mysterious formulas, and is there a point of view from which this definition will be seen as routine and not the result of some intangible inspiration?

The purpose of this paper is to discuss an algebraic theory of which a particular instance gives a formal answer to these questions and to write down some basic definitions and facts. For any (commutative) ring k, we define a k-plethory to be a commutative k-algebra together with a comonad structure on the covariant functor it represents, much as a k-algebra is the same as a k-module that represents a comonad. So, just as a k-algebra is exactly the structure that knows how to act on a k-module, a k-plethory is the structure that knows how to act on a commutative k-algebra. It is not so surprising that this analogy extends further:

Linear/k	Non-linear/k
k-modules M	Commutative k-algebras R
k-k-bimodules N	k-k-birings S
$\mathrm{Hom}_k(N, M)$	$\mathrm{Hom}_{k\text{-alg}}(S, R)$
$N \otimes_k M$	$S \odot_k R$
$k = \otimes$-unit	$k[e] = \odot$-unit
k-algebras A	k-plethories P
A-modules	P-rings
A-A'-bimodules	P-P'-birings

This is explained in Section 1. In fact, as Bergman has informed us, this picture has been known in the universal-algebra community, under quite similar terminology and notation, since Tall and Wraith's paper [19] in 1970. (See also [23,2].) For those familiar with their work, parts of the first sections will be very familiar.

The description of the ring of Witt vectors from this point of view is that there is a $\mathbf{Z}$-plethory Λ_p, and $W(R)$ is simply the Λ_p-ring co-induced from the ring R (which observation allows us to define a Witt ring for any plethory), and so the only thing left is to give a natural construction of Λ_p. This is done by a process we call *amplification* and which is formally similar to performing an affine blow-up in commutative algebra. We will give some idea of this procedure below.

In Section 2, we give some examples of plethories. The most basic is the symmetric algebra $S(A)$ of any cocommutative bialgebra A; in particular, if A is a group algebra $\mathbf{Z}G$, then $S(A)$ is the free polynomial algebra on the set underlying G. These plethories are less interesting because their actions on rings can be described entirely in terms of the original bialgebra A; for example, an action of the plethory $S(\mathbf{Z}G)$ is the same as an action of the group G. But even in this case, there can be more maps between two such plethories than there are between the bialgebras, and in some sense, this is ultimately responsible for existence of Λ_p and hence the p-typical Witt ring.

The ring Λ of symmetric functions in infinitely many variables is a better example. The composition law of Λ is given by the operation known as plethysm in the theory of symmetric functions and is what gives plethories their name. An action of Λ on a ring R is the same as a λ-ring structure on R, and in contrast to plethories of the form $S(A)$, a Λ-action cannot in general be described in terms of a bialgebra action. We also give an explicit description of Λ_p, the plethory responsible for the p-typical Witt ring, in terms of symmetric functions. Of course, this description is really quite close to a standard treatment of the Witt ring and is still a bit unsatisfying. In Section 3, we give explicit examples of P-Witt rings for various plethories P.

In Section 4, we discuss the restriction, induction, and co-induction functors for a morphism $P \to Q$ of plethories, and we state the reconstruction theorem. As always, the content of such a theorem is entirely category theoretic (Beck's theorem). All the same, the result is worth stating:

Theorem. *Let C be a category that has all limits and colimits, let U be a functor from C to the category of rings. If U has both a left and a right adjoint and has the property that a map f in C is an isomorphism if $U(f)$ is, then C is the category of P-rings for a unique k-plethory P, and under this identification, U is the forgetful functor from P-rings to rings.*

In Section 7, we explain amplification, the blow-up-like process we mentioned above. Let $\mathcal{O}$ be a Dedekind domain, for example the ring of integers in a local or global field or the coordinate ring of a smooth curve. Let $\mathfrak{m}$ be an ideal in $\mathcal{O}$, let P be an $\mathcal{O}$-plethory, let Q be an $\mathcal{O}/\mathfrak{m}$-plethory, and let $P \to Q$ be a surjective map of plethories. We say a P-ring R is a *P-deformation of a Q-ring* if it is $\mathfrak{m}$-torsion-free and the action of P on $R/\mathfrak{m}R$ factors through the map $P \to Q$.

Theorem. *There is an $\mathcal{O}$-plethory P' that is universal among those that are equipped with a map from P making them P-deformations of Q-rings. Furthermore, P' has the property that P-deformations of Q-rings are the same as P'-rings that are $\mathfrak{m}$-torsion-free.*

We say P' is the amplification of P along Q.

In Sections 8–11, we define what could be called the linearization of a plethory P. It involves two structures: A_P, the set of elements of P that act additively on any P-ring, and C_P, the cotangent space to the spectrum of P at 0.

Theorem. *Both A_P and C_P are (generally non-commutative) algebras equipped with maps from k, and under certain flatness or splitting hypotheses, the following hold: A_P is a cocommutative twisted k-bialgebra, there is a coaction of A_P on the algebra C_P, and the map $A_P \to C_P$ is A_P-coequivariant.*

We stop short of investigating representations of such linear structures.

If $R \to R'$ is a map of P-rings with kernel I, then all that remains on the conormal module I/I^2 of the action of P is an action of C_P. In particular, C_P acts on the Kähler differentials of any P-ring. In the special case when $P = \Lambda_p$ and $R = W(S)$, for some ring S, this additional structure is essentially a lift of Bloch's Frobenius operator on the de Rham–Witt complex.

The final section of the paper is the reason why the others exist, and we encourage the reader to look at it first. Here, we consider Λ_p and other classical constructions from the point of view of the general theory. For example, we give a satisfying construction of Λ_p: Let $\mathbf{F}_p\langle e\rangle$ be the trivial $\mathbf{F}_p$-plethory; its bialgebra of additive elements has a canonical deformation to a $\mathbf{Z}$-bialgebra, and let P be the free $\mathbf{Z}$-plethory on this. Then Λ_p is the amplification of P along $\mathbf{F}_p\langle e\rangle$. Essentially the same procedure, applied to rings of integers in general number fields, gives at once ramified and twisted generalizations.

An action of this amplification on a p-torsion-free ring R is, essentially by definition, the same as a lift of the Frobenius endomorphism of R/pR. The content of the statement that the Λ_p-ring co-induced by R agrees with the classical $W(R)$ is ultimately just Cartier's Dieudonné–Dwork lemma. Thus it would be accurate to view amplifications as the framework where Joyal's approach to the classical Witt vectors [10] naturally lives.

The last section also has explicit descriptions of the linearizations of Λ_p, Λ, and similar plethories.

On a final note, this paper does not even contain the basics of the theory, and there are still many simple mysteries. For example, the existence of non-linear plethories, those that do not come from (possibly twisted) bialgebras, may be a purely arithmetic phenomenon: we know of no non-linear plethory over a $\mathbf{Q}$-algebra. For a broader example, the category of P-rings is, on the one hand, a generalization of the category of rings and, on the other, an analogue of the category of modules over an algebra. And so it is natural to ask which notions in commutative algebra and algebraic geometry can be generalized to P-rings for general P and, in the other direction, which notions in the theory of modules over algebras have analogues in the theory actions of plethories on rings. It would be quite interesting to see how far these analogies can be taken.

0. Conventions

The word *ring* is short for commutative ring, but we make no commutativity restriction on the word *algebra*. A k-ring is then a commutative k-algebra. All these objects are assumed to be associative and unital, and all morphisms are unital. Ring_k denotes the category of k-rings.

We use the language of coalgebras extensively; Dăscălescu, Năstăsescu, and Raianu's book [5] is more than enough.

For categorical terminology, we refer to Mac Lane's book [14]. In particular, we find it convenient to write $\mathsf{C}(X, Y)$ for the set of morphisms between objects X and Y of a category C.

$\mathbf{N}$ denotes the set $\{0, 1, 2, \ldots\}$.

1. Plethories and the composition product

Let k, k', k'' be rings.

A k-k-biring is a k-ring that represents a functor $\mathsf{Ring}_k \to \mathsf{Ring}_k$. Composition of such functors yields a monoidal structure on the category of k-k-birings. We then define a k-plethory to be a monoid in this category, much as one could define a k-algebra to be a monoid in the category of k-k-bimodules. Finally, the category of k-k-birings acts on the category of k-rings, and we define a P-ring to be a ring together with an action of the k-plethory P.

We spell this out in some detail and give a number of immediate consequences of the definitions. We also give many examples in this section, but they are all trivial, and so the reader may want to look ahead at the more interesting examples in Sections 2 and 3.

1.1. A k-k'-*biring* is a k-ring S, together with a lift of the covariant functor it represents to a functor $\mathsf{Ring}_k \to \mathsf{Ring}_{k'}$. Equivalently, it is the structure on S of a k'-ring object in the opposite category of Ring_k. Or in Grothendieck's terminology, this is the structure on $\operatorname{Spec} S$ of a commutative k'-algebra scheme over $\operatorname{Spec} k$. Explicitly, S is a k-ring with the following additional maps (all of k-rings except (3)):

(1) *coaddition*: a cocommutative coassociative map $\Delta^+: S \to S \otimes_k S$ for which there exists a counit $\varepsilon^+: S \to k$ and an antipode $\sigma: S \to S$,
(2) *comultiplication*: a cocommutative coassociative map $\Delta^\times: S \to S \otimes_k S$ which codistributes over Δ^+ and for which there exists a counit $\varepsilon^\times: S \to k$,
(3) *co-k'-linear structure*: a map $\beta: k' \to \mathsf{Ring}_k(S, k)$ of rings, where the ring structure on $\mathsf{Ring}_k(S, k)$ is given by (1) and (2).

Note that, as usual, ε^+, σ, and $\varepsilon^\times$ are unique if they exist. Also note that omitting axiom (3) leaves us with the notion of k-$\mathbf{Z}$-biring. Finally, in the case of k-plethories, we will take $k = k'$, but at this point it is best to keep the roles separate.

A morphism of k-k'-birings is a map of k-rings which preserves all the structure above. The category of k-k'-birings is denoted $\mathsf{BR}_{k,k'}$. Given a map $k'' \to k'$, we can view a S as a k-k''-biring, which we still denote S, somewhat abusively.

Let ℓ and ℓ' be rings, and let T be a ℓ-ℓ'-biring. A morphism $S \to T$ of birings is the following data: a ring map $k \to \ell$, a ring map $k' \to \ell'$, and a map $\ell \otimes_k S \to T$ of ℓ-k'-birings. The category of birings is denoted BR. When necessary, we will distinguish the structure maps of birings by using subscripts: Δ^+_S, $\varepsilon^\times_S$, and so on. We will also often use without comment the notation $\Delta^+ p = \sum_i p_i^{(1)} \otimes p_i^{(2)}$ and $\Delta^\times p = \sum_i p_i^{[1]} \otimes p_i^{[2]}$.

1.2. Examples.

(1) k itself is the initial k-k'-biring, representing the constant functor giving the zero ring.

(2) Let $k\langle e\rangle$ denote the k-k-biring that represents the identity functor on Ring_k. Thus $k\langle e\rangle$ is canonically the ring $k[e]$ with $\Delta^+(e) = e\otimes 1+1\otimes e$, $\Delta^\times(e) = e\otimes e$, $\beta(c)(e) = c$ (and $\varepsilon^+(e) = 0$, $\varepsilon^\times(e) = 1$, $\sigma(e) = -e$).

(3) If k' is finite, then the collection of set maps $k' \to k$ is naturally a k-k'-biring. The k-ring structure is given by pointwise addition and multiplication, and the coring structure is given by the ring structure on k'. For example, Δ^+ is the composite $k^{k'} \to k^{k'\times k'} = k^{k'} \otimes k^{k'}$, where the first map is given by addition on k'. If k' is not finite, there are topological issues, which could surely be avoided by considering pro-representable functors from Ring_k to $\mathsf{Ring}_{k'}$.

Recall that the action of a k-algebra A on a k-module M can be given in three ways: as a map $A\otimes_k M \to M$, as a map $M \to \mathrm{Mod}_k(A, M)$, or as a map $A \to \mathrm{Mod}_k(M, M)$. In fact, we have the same choices when defining the multiplication map on A itself. The Witt vector approach to operations on rings follows the second, comonadic model, but we will follow the first, monadic one. The third approach encounters the topological problems mentioned in the example above.

We now define the analogue of the tensor product.

1.3. *Functor* $-\odot_{k'} -\colon \mathsf{BR}_{k,k'} \times \mathsf{Ring}_{k'} \to \mathsf{Ring}_k$. Take $S \in \mathsf{BR}_{k,k'}$ and $R \in \mathsf{Ring}_{k'}$. Then $S \odot_{k'} R$ is defined to be the k-ring generated by symbols $s \odot r$, for all $s \in S, r \in R$, subject to the relations (for all $s, s' \in S, r, r' \in R, c \in k'$)

$$ss' \odot r = (s \odot r)(s' \odot r), \quad (s + s') \odot r = (s \odot r) + (s' \odot r), \quad c \odot r = c \tag{1.3.1}$$

and

$$s \odot (r + r') = \Delta^+_S(s)(r, r') := \sum_i (s_i^{(1)} \odot r)(s_i^{(2)} \odot r'),$$

$$s \odot (rr') = \Delta^\times_S(s)(r, r') := \sum_i (s_i^{[1]} \odot r)(s_i^{[2]} \odot r'),$$

$$s \odot c = \beta(c)(s). \tag{1.3.2}$$

This operation is called the *composition product* and is clearly functorial in both R and S.

As in linear algebra, where a tensor $a \otimes b$ reminds us of the formal composition of operators a and b or the formal evaluation of an operator a at b, the symbol $s \odot r$ is intended to remind us of the composition $s \circ r$ of possibly non-linear functions or the formal evaluation of a function s at r. Thus the meaning of (1.3.1) is that ring operations on functions are defined pointwise, and the meaning of (1.3.2) is that there is extra structure on our ring of functions that controls how they respect sums, products, and constant functions. For example, if S is the biring of 1.2(3), the evaluation map $S \odot_{k'} k' \to k$ given by $s \odot r \mapsto s(r)$ is a well-defined ring map.

1.4. Proposition. *Let S be a k-k'-biring. The functor $S \odot_{k'} -$ is the left adjoint of* $\mathsf{Ring}_k(S, -)$.

In other words, for $R_1 \in \mathsf{Ring}_k$, $R_2 \in \mathsf{Ring}_{k'}$ we have

$$\mathsf{Ring}_k(S \odot_{k'} R_2, R_1) = \mathsf{Ring}_{k'}(R_2, \mathsf{Ring}_k(S, R_1)).$$

The proof is completely straightforward. We leave it, as well as the task of specifying the unit and counit of the adjunction, to the reader.

1.5. Examples.
(1) There are natural identifications $S \odot_{k'} k'\langle e\rangle = S$, $k'\langle e\rangle \odot_{k'} R = R$, $S \odot_{k'} k' = k$, and $k \odot_{k'} R = k$.
(2) If $k' \to \ell'$ is a ring map, then $\ell'\langle e\rangle \odot_{k'} R = \ell' \otimes_{k'} R$.
(3) k-ℓ'-biring structures on S compatible with the given k-k'-biring structure are the same, under adjunction, as maps $S \odot_{k'} \ell' \to k$ of k-rings.
(4) If $k \to \ell$ is a ring map, we have $(\ell \otimes_k S) \odot_{k'} R = \ell \otimes_k (S \odot_{k'} R)$.
(5) The composition product distributes over arbitrary tensor products:

$$\left(\bigotimes S_i\right) \odot_{k'} R = \bigotimes (S_i \odot_{k'} R)\,,$$

$$S \odot_{k'} \left(\bigotimes R_i\right) = \bigotimes (S \odot_{k'} R_i)\,.$$

1.6. If R is not only a k'-ring but a k'-k''-biring, then the functor

$$\mathsf{Ring}_k(S \odot_{k'} R, -) = \mathsf{Ring}_{k'}(R, \mathsf{Ring}_k(S, -))$$

naturally takes values in k''-rings, and so $S \odot_{k'} R$ is naturally a k-k''-biring. One can also see this directly in terms of the structure maps Δ^+ and so on by using the fact that the composition product distributes over tensor products. If $k = k' = k''$, the composition product gives a monoidal structure on the category of k-k-birings with unit $k\langle e\rangle = k[e]$ of 1.2. As is generally true with composition or the tensor product of bimodules, this monoidal structure not symmetric.

1.7. Remark. Note that, in contrast to the analogous statement for bimodules, it is generally not true that a k-k''-biring structure on R induces k'-k''-biring structure on the k-ring $\mathsf{Ring}_k(S, R)$.

1.8. A *k-plethory* is a monoid in the category of k-k-birings, that is, it is a biring P equipped with an associative map of birings $\circ\colon P \odot_k P \to P$ and unit $k\langle e\rangle \to P$. For example, $k\langle e\rangle = k[e]$ with $\circ$ taken as in 1.5(1) (that is, composition of polynomials) is a k-plethory. The image of e under the unit map $k\langle e\rangle \to P$ is denoted e (or e_P); together with $\circ$, it gives the set underlying P a monoid structure. The ring k is called the *ring of scalars* of P.

If P' is a k'-plethory, a morphism $P \to P'$ of plethories is a morphism $k \to k'$ plus a morphism $\varphi\colon P \to P'$ of birings which is also a morphism of monoids. This is equivalent to requiring that

$$
\begin{tikzcd}
k'\langle e\rangle \odot_k P \odot_k P \arrow[r, "\varphi\odot 1"] \arrow[dd, "1\odot\circ"] & P' \odot_k P \arrow[r, equal] & P' \odot_{k'} k'\langle e\rangle \odot_k P \arrow[d, "1\odot\varphi"] \\
 & & P' \odot_{k'} P' \arrow[d, "\circ"] \\
k'\langle e\rangle \odot_k P \arrow[r, equal] & k' \otimes_k P \arrow[r, "\varphi"] & P'
\end{tikzcd}
$$

be a commutative diagram of k'-k-birings. If $k = k'$, the diagram simplifies to the obvious one. If we are already given a map $k \to k'$, then we will always assume the map of scalars is the same as the given map. It is easy to see that $k\langle e\rangle$ is the initial k-plethory and $\mathbf{Z}\langle e\rangle$ is the initial plethory.

1.9. A (*left*) *action* of P on a k-ring R is defined as usual in the theory of monoidal categories; in this case it means a map $\circ\colon P \odot R \to R$ such that $(\alpha\circ\beta)\circ r = \alpha\circ(\beta\circ r)$ and $e\circ r = r$ for all $\alpha, \beta \in P, r \in R$. We also denote $\alpha\circ r$ by $\alpha(r)$. A *P-ring* is a k-ring equipped with an action of P. (There is no danger of a conflict in terminology with a ring equipped with a ring map from P because we never use such structures in this paper.) A morphism of P-rings is a map of rings that makes the obvious diagram commute; equivalently, it is a map of rings that is P-equivariant as a map of sets acted on by the monoid $(P, \circ)$. The category of P-rings is denoted Ring_P.

If S is a k-k'-biring, we say *P acts on S* as a *k-k'-biring* if $\circ\colon P \odot S \to S$ is a map of k-k'-birings. Such an action is the same as a functorial collection of k'-ring structures on the sets $\mathsf{Ring}_P(S, R)$ such that the maps $\mathsf{Ring}_P(S, R) \hookrightarrow \mathsf{Ring}_k(S, R)$ are maps of k'-rings.

A *right action* of a k'-plethory P' on a k-k'-biring is a map $\circ\colon R \odot_{k'} P' \to R$ of k-k'-birings compatible with $\circ$ and e in the obvious way. A map of right P'-rings is P'-equivariant map of k-k'-birings. A P-P'-biring is a k-k'-biring equipped with a left action of P as a k-k' biring and a commuting right action of P'. The category of P-P'-birings is denoted $\mathsf{BR}_{P,P'}$, morphisms being maps of birings that are both

P-equivariant and P'-equivariant. A P-P'-biring is the same as a represented functor $\mathsf{Ring}_P \to \mathsf{Ring}_{P'}$.

1.10. A k-plethory structure on a k-k-biring P is the same as a monad structure on the functor $P \odot_k -$ and, by adjunction, also the same as a comonad structure on the functor $\mathsf{Ring}_k(P, -)$. An action of P on R is the same as the structure on R of an algebra over the monad or a coalgebra over the comonad.

Thus Ring_P has all limits and colimits, the forgetful functor $U: \mathsf{Ring}_P \to \mathsf{Ring}_k$ preserves them, and the functors $P \odot_k -$ and $\mathsf{Ring}_k(P, -)$ lift to give left and, respectively, right adjoints to U. (These functors could well be called restriction, induction, and co-induction for the map $k\langle e\rangle \to P$. We postpone the treatment of these functors for general maps of plethories until section four.) In particular, the underlying k-ring of a (co)limit of P-rings is the (co)limit in that category and there exists a unique compatible P-ring structure on it. We give a converse to all this in Section 4.

We often denote the functor $\mathsf{Ring}_k(P, -)$ by $W_P(-)$ and call the P-ring $W_P(R)$ the *P-Witt ring* of R. The reason for this terminology will be made clear in Section 3.

1.11. Examples.

(1) If k is finite, the biring of set maps $k \to k$ is a k-plethory, with $\circ$ given by composition of functions. In particular, 0 is a plethory over the ring 0. It is the terminal plethory, and of course the only 0-ring is 0.

(2) A plethory P clearly acts on itself on the left (and also the right). It is in fact the free P-ring on one element: morphisms in Ring_P from P to another object are the same as elements of the underlying ring, a map $\varphi: P \to R$ corresponding to the element $\varphi(e)$ in R, and an element $r \in R$ corresponding to the map $\alpha \mapsto \alpha(r)$. The morphisms $P \to k$ corresponding to $r = 0$ and $r = 1$ are ε^+ and $\varepsilon^\times$. More generally, the morphism $P \to k$ corresponding to $c \in k$ is $\beta(c)$.

(3) The identification $P \odot_k k = k$ is an action of P on k, and if R is any P-ring, the structure map $k \to R$ is a map of P-rings simply by the third relation of (1.3.2). Therefore, k is the initial P-ring. Similarly, the identification $k \odot_k P = k$ gives k the structure of a P-P-biring, and it is the initial P-P-biring.

(4) If k' is a P-ring, the natural k'-map

$$(k' \otimes_k P) \odot_k k' = k' \otimes_k (P \odot_k k') \to k'$$

gives (by 1.5) $k' \otimes_k P$ the structure of a k'-k'-biring. We will see below that $k' \otimes_k P$ even has a natural k'-plethory structure.

1.12. Proposition. *Let P be a k-plethory. Then the k-ring morphisms Δ_P^+, $\Delta_P^\times$, ε_P^+, and $\varepsilon_P^\times$ are in fact P-ring morphisms. For any $A \in \mathsf{Ring}_P$, the unit $\eta_A: k \to A$ and multiplication $m_A: A \otimes_k A \to A$ are P-ring morphisms.*

Proof. The unit and counits were discussed in 1.11(3) and (2). Multiplication is the coproduct of the identity with itself.

By 1.11(2), the P-ring P represents the forgetful functor U' from Ring_P to the category of sets and $P \otimes_k P$ represents the functor $U' \times U'$. But these factor through the category of rings, and so there are natural transformations $U' \times U' \to U'$, one for addition and one for multiplication. Thus there are maps $P \to P \otimes_k P$ in Ring_P. The one for addition is the map that sends e to $1 \otimes e + e \otimes 1$, and thus sends α to $\Delta^+(\alpha)(1 \otimes e, e \otimes 1) = \Delta^+(\alpha)$. Similarly, the one for multiplication is $\Delta^\times$. $\square$

1.13. *Base change of plethories.* If k' is a P-ring, then the k'-k-biring $k' \otimes_k P$ has a k'-k'-biring structure (1.11). Even further, the k'-ring map (using 1.5(4))

$$(k' \otimes_k P) \odot_k (k' \otimes_k P) = k' \otimes_k (P \odot_k (k' \otimes_k P)) \xrightarrow{1 \otimes \circ} k' \otimes_k (k' \otimes_k P) \longrightarrow k' \otimes_k P$$

descends to a map

$$(k' \otimes_k P) \odot_{k'} (k' \otimes_k P) \longrightarrow k' \otimes_k P,$$

which gives $k' \otimes_k P$ the structure of a k'-plethory.

Conversely, if $k' \otimes P$ is a k'-plethory, then P acts on k' by way of $k' \otimes P$. Note that not only does the plethory structure on $k' \otimes P$ depend on the action of P on k', there may not exist even one such action. For example, there is no action of the $\mathbf{Z}$-plethory Λ_p (of 2.13) on $\mathbf{F}_p$.

We leave it as an exercise to show that a $k' \otimes P$-action on a k'-ring R is the same as a P-action on the underlying k-ring compatible with the given action on k'.

2. Examples of plethories

Before continuing with the theory, let us give some basic examples of plethories.

2.1. *Free plethory on a biring.* Let k be a ring, and let S be a k-k-biring. There is a plethystic analogue of the tensor algebra: a k-plethory Q, with a k-k-biring map $S \to Q$, which is initial in the category of such plethories.

Put

$$Q = \bigotimes_{n \geqslant 0} S^{\odot n}.$$

The system of maps

$$S^{\odot i} \odot S^{\odot j} \longrightarrow S^{\odot (i+j)}$$

$$(s_1 \odot \cdots \odot s_i) \odot (t_1 \odot \cdots \odot t_j) \mapsto s_1 \odot \cdots \odot s_i \odot t_1 \odot \cdots \odot t_j$$

induces a map

$$Q \odot Q = \bigotimes_{i,j} S^{\odot i} \odot S^{\odot j} \longrightarrow \bigotimes_{n} S^{\odot n} = Q,$$

which is clearly associative. This gives Q the structure of a k-plethory with a map $k\langle e\rangle = S^{\odot 0} \to Q$ of k-plethories.

A Q-action on a ring R is then the same as a map $S \odot R \to R$ of rings.

2.2. *Free plethory on a cocommutative bialgebra.* First, let A be a cocommutative coalgebra over k; denote its comultiplication map by Δ and its counit by ε. The symmetric algebra $S(A)$ of A, viewed as a k-module, is of course a k-ring, but the following gives it the structure of a k-k-biring:

Coadditive structure: The coaddition map Δ^{+} is the one induced by the linear map

$$A \longrightarrow S(A) \otimes S(A), \quad a \mapsto a \otimes 1 + 1 \otimes a.$$

The additive counit $\varepsilon^{+}: S(A) \to k$ is the map induced by the zero map $A \to k$.

Comultiplicative structure: $\Delta^{\times}$ is the map induced by the linear map

$$A \xrightarrow{\Delta} A \otimes A \longrightarrow S(A) \otimes S(A),$$

where the right map is the tensor square of the canonical inclusion. The multiplicative counit $\varepsilon^{\times}: S(A) \to k$ is the composite map

$$S(A) \xrightarrow{S(\varepsilon)} S(k) = k\langle e\rangle \xrightarrow{\varepsilon^{\times}_{k\langle e\rangle}} k.$$

Co-k-linear structure: The map

$$S(A) \odot_{\mathbf{Z}} k \longrightarrow k\langle e\rangle \odot_{\mathbf{Z}} k \longrightarrow k\langle e\rangle \odot_{k} k = k$$

gives $S(A)$ a k-k-biring structure by 1.5.

2.3. *Isomorphism $S(A) \odot S(B) \to S(A \otimes B)$ of k-k-birings.* Let B be another cocommutative k-coalgebra, and let R be a k-ring. Then we have

$$\begin{aligned} Ring_k(S(A) \odot S(B), R) &= Ring_k(S(B), \mathsf{Ring}_k(S(A), R)) \\ &= \mathrm{Mod}_k(B, \mathrm{Mod}_k(A, R)) = \mathrm{Mod}_k(A \otimes B, R) \\ &= Ring_k(S(A \otimes B), R) \end{aligned}$$

and hence a natural isomorphism $S(A) \odot S(B) \cong S(A \otimes B)$ of k-rings. Explicitly, $[a] \odot [b]$ corresponds to $[a \otimes b]$, where $[a]$ denotes the image of a under the natural inclusion $A \to S(A)$ and likewise for $[b]$. We leave the task of showing this is a map of k-k-birings to the reader.

2.4. It follows that the comultiplication and the counit induce maps

$$S(A) \longrightarrow S(A) \odot S(A),$$

$$S(A) \longrightarrow k\langle e \rangle$$

that give $S(A)$ the structure of a commutative comonoid in $\mathsf{BR}_{k,k}$.

2.5. Now suppose A is a bialgebra, that is, A is equipped with maps

$$A \otimes A \longrightarrow A,$$

$$k \longrightarrow A$$

of k-coalgebras making A a monoid in the category of k-coalgebras. By the discussion above, this makes $S(A)$ a monoid in the category of cocommutative comonoids in $\mathsf{BR}_{k,k}$. It is in particular a k-plethory. (It could reasonably be called a cocommutative bimonoid in $\mathsf{BR}_{k,k}$—its additional structure is the analogue of the structure added to an algebra to make it a cocommutative bialgebra—but because $\odot$ is not a symmetric operation on all of $\mathsf{BR}_{k,k}$, this terminology could be confusing.)

2.6. Remark. Given a k-ring R, an action of the plethory $S(A)$ on R is the same as an action of the bialgebra A on R. We leave the precise formulation and proof of this to the reader. It may be worth noting that any k-ring admits an $S(A)$-action in a trivial way. This is true by the previous remark or by using the natural map $S(A) \to k\langle e \rangle$ of k-plethories. It is false for general plethories.

2.7. Examples.

(1) If A is the group algebra kG of a group (or monoid) G, then $S(A)$ is the free polynomial algebra on the set underlying G. For any $g \in G$, the corresponding element in $S(A)$ is "ring-like": $\Delta^+(g) = g \otimes 1 + 1 \otimes g$ and $\Delta^\times(g) = g \otimes g$. An action of the plethory $S(A)$ on a ring R is the same as an action of G on R.

(2) Let $\mathfrak{g}$ be a Lie algebra over k, and let A be its universal enveloping algebra. Then for all $x \in \mathfrak{g}$, the corresponding element $x \in S(A)$ is "derivation-like": $\Delta^+(x) = x \otimes 1 + 1 \otimes x$ and $\Delta^\times(x) = x \otimes e + e \otimes x$. If $\mathfrak{g}$ is the one-dimensional Lie algebra spanned by an element d, then $S(A) = k[d^{\circ \mathbf{N}}] := k[e, d, d \circ d, \ldots]$, and $S(A)$-rings are the same as k-rings equipped with a derivation.

2.8. Remark. Because of the identification $S(A) \odot_k S(B) \to S(A \otimes B)$, there is a natural isomorphism $S(A) \odot_k S(B) \to S(B) \odot_k S(A)$ of k-k-birings given by the canonical interchange map on the tensor product. Explicitly, it exchanges $[a] \odot [b]$ and $[b] \odot [a]$, where $a \in A, b \in B$. There is no functorial map $S \odot T \to T \odot S$ for k-k-birings S and T that agrees with the previous map when S and T come from bialgebras. For example, take $S = \mathbf{Z}[d^{\circ \mathbf{N}}]$ and $T = \Lambda_p$ below.

2.9. *Hopf algebras.* An antipode $s: A \to A$ gives a map $S(A) \to S(A)$ of k-k-birings, making $S(A)$ what could be called a cocommutative Hopf monoid in $\mathsf{BR}_{k,k}$.

2.10. *Symmetric functions and λ-rings.* Let Λ be the ring of symmetric functions in countably many variables, i.e., writing Λ_n for the sub-graded-ring of $\mathbf{Z}[x_1, \ldots, x_n]$ ($\deg x_i = 1$) of elements invariant under the obvious action of the n-th symmetric group, we let Λ be the inverse limit of

$$\cdots \longrightarrow \Lambda_n \longrightarrow \Lambda_{n-1} \longrightarrow \cdots .$$

in the category of graded rings. The map above sends x_n to 0 and sends any other x_i to x_i. Of course, Λ is the free polynomial algebra on the elementary symmetric functions [15, I.2], but there are many other free generating sets, and making this or any other particular choice would leave us with the usual formulaic mess in the theory of λ-rings and Witt vectors.

The ring Λ naturally has the structure of a plethory over $\mathbf{Z}$. Because all the structure maps are already described at various points in the second edition of MacDonald [15], we give only the briefest descriptions here:

Coadditive structure [15, I.5 ex. 25]: For $f \in \Lambda$, consider the function

$$\Delta^+(f) = f(x_1 \otimes 1, 1 \otimes x_1, x_2 \otimes 1, 1 \otimes x_2, \ldots)$$

in the variables $x_i \otimes x_j$, $(i, j \geqslant 1)$. It is symmetric in both factors, and so Δ^+ is a ring map $\Lambda \to \Lambda \otimes_{\mathbf{Z}} \Lambda$. The counit $\varepsilon^+: \Lambda \to k$ sends f to $f(0, 0, \ldots)$.

Comultiplicative structure [15, I.7 ex. 20]: Similarly, consider the function

$$\Delta^\times(f) = f(\ldots, x_i \otimes x_j, \ldots)$$

in the variables $x_i \otimes x_j$. As before, it is symmetric in both factors, and so $\Delta^\times$ is a map $\Lambda \to \Lambda \otimes_{\mathbf{Z}} \Lambda$. The counit $\varepsilon^\times : \Lambda \to k$ sends f to $f(1, 0, 0, \ldots)$.

Monoid structure [15, I.8]: For $f, g \in \Lambda$, the operation known as plethysm defines $f \circ g$: Suppose g has only non-negative coefficients, and write g as a sum of monomials with coefficient 1 in the variables x_i. Then $f \circ g$ is the symmetric function obtained by substituting these monomials into the arguments $x_1, x_2, \cdots$ of f. This gives a monoid structure with identity $x_1 + x_2 + \cdots$ on the set of elements with non-negative coefficients, and this extends to a unique $\mathbf{Z}$-plethory structure on all of Λ.

2.11. Remark. By the theorem of elementary symmetric functions [15, I2.4], we have

$$\Lambda = \mathbf{Z}[\lambda_1, \lambda_2, \ldots],$$

where $\lambda_1 = x_1 + x_2 + \cdots, \lambda_2 = x_1x_2 + x_1x_3 + x_2x_3 + \cdots, \ldots$ are the elementary symmetric functions. Any Λ-ring R therefore has unary operations $\lambda_1, \lambda_2, \ldots$. It is an exercise in definitions to show that in this way, a Λ-ring structure on a ring R is the same as a λ-ring structure (which, in Grothendieck's original terminology [1], is called a special λ-ring structure). This was in fact one of the principal examples in Tall and Wraith's paper [19].

Let ψ_n denote the nth Adams operation:

$$\psi_n = x_1^n + x_2^n + \cdots \ .$$

The elements $w_1, w_2, \ldots$ of Λ determined by the relations

$$\psi_n = \sum_{d|n} d w_d^{n/d} \quad \text{for all } n \in \mathbf{N} \tag{2.11.1}$$

also form a free generating set. This is easy to check using the following identity:

$$\sum_{n \geqslant 0} (-1)^n \lambda_n t^n = \prod_{i \geqslant 1} (1 - x_i t) = \exp\left(- \sum_{n \geqslant 1} \frac{1}{n} \psi_n t^n \right) = \prod_{n \geqslant 1} (1 - w_n t^n).$$

The w_i are responsible for the Witt components, as we will see in the next section.

2.12. Remark. There is also a description of Λ in terms of the representations of the symmetric groups [15, I.7]. Let R_n denote the representation ring of S_n, the symmetric group on n letters. The maps $S_n \times S_m \to S_{n+m}$, $S_n \to S_n \times S_n$, and $S_n \wr S_m = S_n \ltimes S_m^n \to S_{mn}$ induce maps between the R_n by restriction and induction, and these make up a plethory structure on $\bigoplus_{n \geqslant 0} R_n$ agreeing with that on Λ. This is one natural way to view Λ when studying its action on Grothendieck groups (see, e.g. [6]).

We do not yet know if similar constructions in other areas of representation theory also yield plethories.

2.13. *p-typical symmetric functions*. Let p be a prime number, and set $F = \psi_p$. Then $\mathbf{Z}\langle F \rangle := \mathbf{Z}[e, F, F \circ F, \ldots]$ is a subring of Λ, and because F is ring-like, it is actually a sub-$\mathbf{Z}$-plethory. It is also the free plethory on the bialgebra associated to the monoid $\mathbf{N}$. We will denote it Ψ_p, and we will see later that it accounts for the ghost components of the p-typical Witt vectors.

Now let Λ_p be the subring of Λ consisting of elements f for which there exists an $i \in \mathbf{N}$ such that $p^i f \in \Psi_p$. Then Λ_p is a sub-$\mathbf{Z}$-plethory of Λ, and is what we call the plethory of *p-typical symmetric functions*.

For all $n \in \mathbf{N}$, let $\theta_n = w_{p^n}$. Then (2.11.1) becomes

$$F^{\circ n} = \theta_0^{p^n} + \cdots + p^n \theta_n \tag{2.13.1}$$

and therefore $\theta_0, \theta_1, \ldots$ lie in Λ_p. Conversely, because we have

$$\Lambda = \mathbf{Z}[\theta_0, \theta_1, \ldots][w_n \,|\, n \text{ is not a power of } p],$$

we see $\Lambda_p = \mathbf{Z}[\theta_0, \theta_1, \ldots]$.

2.14. *Binomial plethory.* Because Λ is a $\mathbf{Z}$-plethory, the ring $\mathbf{Z}$ of integers is a Λ-ring. The ideal in Λ of elements that act as the constant function 0 is generated by the set $\{\psi_n - e \,|\, n \geqslant 1\}$. The quotient ring is still a plethory, and an action of it on a ring R is the same as giving R the structure of a Λ-ring whose Adams operations are the identity. This has been shown by Jesse Elliott (unpublished) to be the same as a binomial λ-ring structure [11, p. 9] on R.

This plethory can also be interpreted as the set of functions $\mathbf{Z} \to \mathbf{Z}$ that can be expressed as polynomials with rational coefficients [2].

3. Examples of Witt rings

Let k be a ring. Recall that if P is a k-plethory, then $W_P(R)$ denotes the P-ring $\mathsf{Ring}_k(P, R)$. Because W_P is the right adjoint of the forgetful functor from P-rings to rings, there is a natural map $W_P \to W_P(W_P(R))$, which in the case of the classical plethories is sometimes called the Artin–Hasse map.

3.1. *Bialgebras.* Let P be the free k-plethory (2.2) on a cocommutative k-bialgebra A. Then we have $W_P(B) = \mathrm{Mod}_k(A, B)$. If A is finitely generated as a k-module, $W_P(B)$ is just $B \otimes_k A^*$, where A^* denotes the dual bialgebra $\mathrm{Mod}_k(A, k)$. We leave it to the reader to verify that, in this case, the map $W_P(B) \longrightarrow W_P(W_P(B))$ is nothing but the comultiplication map on this bialgebra. For example, if A is the group algebra of a finite group G, then we have $W_P(B) = B^G$ and the map above is the map $B^G \to B^{G\times G} = B^G \otimes_B B^G$ induced by the multiplication on G.

3.2. *Symmetric functions.* Because $\Lambda = \mathbf{Z}[\lambda_1, \ldots]$, the set $W_\Lambda(B)$ is just $\prod_{n>0} B$, and it is easy to check that, as a group, we have $W_\Lambda(B) = 1 + xB[[x]]$, where the group operation on the right is multiplication of power series. It is also true that if $1 + xB[[x]]$ is given a Λ-ring structure as in [1, 1.1], then the identification above is an isomorphism of Λ-rings, i.e., $W_\Lambda(B)$ is the Λ-ring of "big" Witt vectors. The proof of this is very straightforward but involves, of course, the somewhat unpleasant definition of the Λ-ring structure on $1+xB[[x]]$. Because the whole point of this paper is to move away from such things, we will leave the argument to the reader. The generating set $\{w_1, w_2, \ldots\}$ of 2.11 allows us to view an element of $W_\Lambda(B)$ as a ("big") Witt vector in the traditional sense [8, 17.1.15]. Under this identification, the map $W_\Lambda(B) \longrightarrow W_\Lambda(W_\Lambda(B))$ agrees with the usual Artin–Hasse map [8, 17.6].

If Ψ denotes the sub-plethory $\mathbf{Z}[\psi_n \,|\, n \geqslant 1]$ of Λ, then $W_\Psi(B)$ is just $\prod_{n>0} B$ as a ring, and under this identification, the map $W_\Lambda(B) \to W_\Psi(B)$ is the ghost-component map.

Some early references to the big Witt vectors are Cartier [4] and Witt ([12] or [22, pp. 157–163]).

3.3. *p-typical symmetric functions.* Because $\Lambda_p = \mathbf{Z}[\theta_0, \ldots]$, the set $W_{\Lambda_p}(B) = \mathsf{Ring}_{\mathbf{Z}}(\Lambda_p, B)$ is naturally bijective with $B^{\mathbf{N}}$. If we view $B^{\mathbf{N}}$ as the set underlying the ring of p-typical Witt vectors [21], [8, 17.1.15], then this bijection is an isomorphism of rings. One can write down the corresponding Λ_p-action on $B^{\mathbf{N}}$, and we recover the p-typical Artin–Hasse map as we did above. Also as above, if Ψ_p denotes the plethory $\mathbf{Z}[\psi_p^{\circ \mathbf{N}}]$, then the natural map $W_{\Lambda_p}(B) \to W_{\Psi_p}(B)$ is the p-typical ghost-component map.

The Teichmüller lift can be constructed by considering the monoid algebra $\mathbf{Z}B$ on the multiplicative monoid underlying B. The ring $\mathbf{Z}B$ has no additive p-torsion, and the map $F\colon [b] \mapsto [b^p] = [b]^p$ ($[-]$ denoting the multiplicative map $B \to \mathbf{Z}B$) reduces to the Frobenius map modulo p. The ring $\mathbf{Z}[B]$ therefore (3.4) admits a unique Λ_p-ring structure where F is the above map. The canonical ring map $\mathbf{Z}B \to B$ then induces by adjointness a map $\mathbf{Z}B \to W_{\Lambda_p}(B)$. In the standard description, it is $[b] \mapsto (b, 0, 0, \ldots)$, which is of course the Teichmüller lift of b.

The following lemma implies that a Λ_p-ring is the same as what Joyal calls a δ-ring. (A comonadic version of this statement is stated quite clearly in Joyal [10]; we include it only because we will use it later.)

3.4. Lemma. *The R be a p-torsion-free ring. Given an action of Λ_p on R, the element F gives an endomorphism of R such that $F(x) \equiv x^p \bmod pR$. This is a bijection from the set of actions of Λ_p on R to the set of lifts of the Frobenius endomorphism of R/pR.*

Proof. Because R is p-torsion-free, (2.13.1) implies that any action of Λ_p is determined by the endomorphism F, and so we need only show every Frobenius lift comes from some action of Λ_p.

Given a Frobenius lift $f\colon R \to R$, Cartier's Dieudonné–Dwork lemma [13, VII Section 4] states there is a ring map $R \to W_{\Lambda_p}(R)$ such that the composite $R \to W_{\Lambda_p}(R) \to W_{\Psi_p}(R)$ sends r to $(r, f(r), f(f(r)), \ldots)$. This gives a map $\Lambda_p \odot R \to R$; to show it is an action we need only check it is associative. Because R is p-torsion-free it suffices to check the induced map of $\Psi_p \odot R \to R$ is an action. But the Dieudonné–Dwork lemma implies this map sends $F^{\circ i} \odot r$ to $f^{\circ i}(r)$, which is clearly associative. $\square$

4. Reconstruction and recognition

In preparation for the reconstruction theorem, we generalize the notions of biring and plethory from Ring_k to Ring_P for non-trivial plethories P. This gives us P-P'-birings and P-plethories, which reduce to k-k'-birings and k-plethories when $P = k\langle e\rangle$ and $P' = k'\langle e\rangle$.

Let P be a k-plethory and P' a k'-plethory, where k and k' are arbitrary rings.

4.1. *Functor* $-\odot_{P'} -\colon \mathsf{BR}_{P,P'} \times \mathsf{Ring}_{P'} \to \mathsf{Ring}_P$. Take $S \in \mathsf{BR}_{P,P'}$ and $R \in \mathsf{Ring}_{P'}$. Then $S \odot_{P'} R$ is defined to be the coequalizer of the maps of P-rings

$$S \odot_{k'} P' \odot_{k'} R \rightrightarrows S \odot_{k'} R$$

$$s \odot \alpha \odot r \mapsto (s \circ \alpha) \odot r$$

$$s \odot \alpha \odot r \mapsto s \odot (\alpha \circ r).$$

4.2. Lemma. *Let S be a P-P'-biring. Then the functor $S \odot_{P'} -\colon \mathsf{Ring}_{P'} \to \mathsf{Ring}_P$ is the left adjoint of the functor $\mathsf{Ring}_P(S,-)$.*

We leave the proof to the reader.

4.3. Proposition. *Let $P \to Q$ be a map of plethories. Then the restriction functor $\mathsf{Ring}_Q \to \mathsf{Ring}_P$ preserves limits and coequalizers and has a left adjoint* (“*induction*”) *$Q \odot_P -$. If the map $P \to Q$ is an isomorphism on scalars, it has a right adjoint* (“*co-induction*”) *$\mathsf{Ring}_P(Q,-)$ and preserves all colimits.*

Proof. Because Q is a Q-P-biring, $Q \odot_P -$ is left adjoint (by 4.2) to $\mathsf{Ring}_Q(Q,-)$, which is the forgetful functor $\mathsf{Ring}_Q \to \mathsf{Ring}_P$. If $P \to Q$ is a map of k-plethories, Q is a P-Q-biring, so $\mathsf{Ring}_P(Q,-)$ is right adjoint to $Q \odot_Q -$, the forgetful functor. It follows that the forgetful functor preserves limits and, when the rings of scalars agree, colimits. It remains to show it always preserves coequalizers.

Consider the commutative diagram of forgetful functors

$$\begin{array}{ccc} \mathsf{Ring}_Q & \longrightarrow & \mathsf{Ring}_{k_Q} \\ \downarrow & & \downarrow \\ \mathsf{Ring}_P & \longrightarrow & \mathsf{Ring}_{k_P}. \end{array}$$

The upper functor preserves colimits, and the right-hand functor preserves coequalizers. The lower functor reflects isomorphisms and preserves colimits. It then follows that the left-hand functor preserves coequalizers. $\square$

4.4. Remark. If $k_P \to k_Q$ is not an isomorphism, ε^+ will fail to descend. Thus, Q will not be a k_P-k_Q-biring, let alone a P-Q-biring.

4.5. A *P-plethory* is defined to be a plethory Q equipped with a map $P \to Q$ of plethories which is an isomorphism on scalars. A morphism $Q \to Q'$ of P-plethories is a morphism of plethories commuting with the maps from P.

4.6. Proposition. *$- \odot_P -$ makes $\mathsf{BR}_{P,P}$ into a monoidal category with unit object P. Monoids in this category are the same as P-plethories. An action of such a monoid Q on a P-ring is the same as an action of Q on the underlying k-ring such that the action of Q restricted to P is the given one.*

Proof. The first statement requires no proof. Given a monoid Q, the structure maps give map $Q \odot_k Q \to Q \odot_P Q \to Q$ and $P \to Q$ making it a k-plethory. Conversely, a map $P \to Q$ of k-plethories makes Q a P-P-biring and the associativity condition $Q \odot_k Q \odot_k Q \rightrightarrows Q \odot_k Q \to Q$ implies that $Q \odot_k P \odot_k Q \rightrightarrows Q \odot_k Q \to Q$ commutes, so composition descends to $Q \odot_P Q \to Q$.

Similarly, an action of Q on the underlying k-ring of a P-ring A is a map $Q \odot_k A \to A$, and it descends to a P-action $Q \odot_P A \to A$ because $Q \odot_k P \odot_k A \rightrightarrows Q \odot_k A \to A$ commutes. $\square$

4.7. Now let C be a category that has all limits and colimits, and let $U: \mathsf{C} \to \mathsf{Ring}_P$ be a functor that has a left adjoint F. We also assume U reflects isomorphisms, that is, a morphism f is an isomorphism if and only if $U(f)$ is an isomorphism. Set $Q = UF(P)$. Let U' be the composite of U with the forgetful functor from Ring_P to the category of sets.

4.8. *k-Plethory structure on Q when U has a right adjoint.* Suppose U has a right adjoint W. The functor UW is represented by Q: $UW(A) = \mathsf{Ring}_P(P, UW(A)) = \mathsf{Ring}_P(UF(P), A)$, and this gives Q the structure of a P-P-biring (1.9). The composite UW of adjoints is a comonad, and so its adjoint $Q \odot_P -$ is a monad. By 4.6, Q is a k-plethory with a map $P \to Q$.

Given an object A of C, the adjunction gives an action of $UF(-) = Q \odot -$ on $U(A)$, and hence we have a functor $\mathsf{C} \to \mathsf{Ring}_Q$ between categories over Ring_P.

4.9. Theorem. *If U has a right adjoint W, then the functor $\mathsf{C} \to \mathsf{Ring}_Q$ is an equivalence of categories over Ring_P.*

Proof. Beck's theorem [14]. $\square$

4.10. Let k' be the P-ring $UF(k)$, and let P' be the k'-plethory $k' \otimes_k P$. Because $F(k)$ is the initial object, U factors as a functor $U': \mathsf{C} \to \mathsf{Ring}_{P'}$ followed by the forgetful functor $V: \mathsf{Ring}_{P'} \to \mathsf{Ring}_P$. The functor U' has a left adjoint F' given by descent: if A is a P'-ring, then $FV(A)$ has two maps from $F(k') = FUF(k)$, one from applying

FV to the initial map $k' \to A$ and the other given by the composite

$$FUF(k) \to F(k) \to FV(A),$$

where the first map is the adjunction and the second is the initial map. Let $F'(A)$ denote the coequalizer of $F(k') \rightrightarrows FV(A)$.

4.11. Theorem. *If $P \to Q$ is a map of plethories and U is the forgetful functor $\mathsf{Ring}_Q \to \mathsf{Ring}_P$, then U' of* 4.10 *has a right adjoint. Conversely, suppose $U'\colon \mathsf{C} \to \mathsf{Ring}_P$ has a right adjoint, and let Q be the k'-plethory $U'F'(P')$ of* 4.8. *Then the functor $\mathsf{C} \to \mathsf{Ring}_Q$ is an equivalence of categories over $\mathsf{Ring}_{P'}$.*

Proof. Apply 4.9 to U'. $\square$

4.12. Remark. In practice, it is quite easy to check the existence of F and W' using Freyd's theorem from category theory.

5. *P*-ideals

Let P be a k-plethory, and let P_+ denote the kernel of $\varepsilon^+\colon P \to k$.

5.1. An ideal I in a P-ring R is called a (*left*) *P-ideal* if there exists an action of P on R/I such that the map $R \to R/I$ of rings is a map of P-rings. If such an action exists, it is unique, and so being a P-ideal is a property of, rather than a structure on, a subset of R.

5.2. Proposition. *Let I be an ideal in a P-ring R. Then the following are equivalent*:

(1) *I is a P-ideal*;
(2) *I is the kernel of a morphism of P-rings*;
(3) $P_+ \circ I \subseteq I$;
(4) *I is generated by a set X such that $P_+ \circ X \subseteq I$.*

The proof is in 5.6.

Given any subset X of P, it is therefore reasonable to call the ideal generated by $P_+ \circ X$ the *P-ideal generated by X*.

5.3. Elements of $P \otimes P$ give binary operations on any P-ring R by $(\alpha \otimes \beta)(r, s) = \alpha(r)\beta(s)$ and extending linearly.

5.4. Lemma. *Let R be a P-ring, I an ideal in R and X a subset of R. Assume that for all $x \in X$ and $f \in P_+$, we have $f(x) \in I$. Then for all $t \in P \otimes P_+$ and all $(r, i) \in R \times I$, we have $t(r, i) \in I$.*

Proof. Since $t \in P \otimes P_+$, it may be expressed as $t = \sum t'_j \otimes t''_j$ with $t''_j \in P_+$, so that t''_j preserves I. Then for $(r, i) \in R \times I$, $t(r, i) = \sum t'_j(r) t''_j(i) \in I$. $\square$

Typical applications will use $X = I$, a P-ideal.

5.5. Lemma. *Let S be a k-$\mathbf{Z}$-biring. Then $\Delta^+(S_+)$ is contained in $S_+ \otimes S + S \otimes S_+$, and $\Delta^\times(S_+)$ is contained in $S_+ \otimes S_+$.*

Proof. S is a ring object in the opposite of Ring_k; the ring identity $0 + 0 = 0$ translates into the identity $(\varepsilon^+ \otimes \varepsilon^+) \circ \Delta^+ = \varepsilon^+$, which is clearly equivalent to the first statement. The second statement is similarly just a coalgebraic translation of a ring identity. Let W denote the ring object corresponding to S in the opposite category. Then the commutativity of the following two diagrams is equivalent:

$$\begin{tikzcd} W & W \times W \arrow[l, "\times"'] \\ 0 \arrow[u] & W \amalg W. \arrow[l] \arrow[u, "\mathrm{id}\times 0 \amalg 0 \times \mathrm{id}"'] \end{tikzcd} \qquad \begin{tikzcd} S \arrow[r, "\Delta^\times"] \arrow[d, "\varepsilon^+"] & S \otimes S \arrow[d, "\mathrm{id}\otimes\varepsilon^+ \times \varepsilon^+\otimes\mathrm{id}"] \\ k \arrow[r] & S \times S \end{tikzcd}$$

But the commutativity of the first is just a restatement of the ring identity $0 \cdot x = x \cdot 0 = 0$. We therefore have

$$\Delta^\times(S_+) \subseteq \ker\Big(S \otimes S \to S \times S\Big) = S_+ \otimes S_+. \qquad \square$$

5.6. Proof of 5.2. (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (4) are clear.

(2) $\Rightarrow$ (3): P_+ preserves the set $\{0\}$ in k and, thus, in any P-ring; it therefore must preserve its preimage under a morphism of P-rings.

(3) $\Rightarrow$ (1): If I is preserved by P_+, we must put a P-ring structure on R/I so that $R \to R/I$ is a morphism of P-rings. The action must be $p(r + I) = p(r) + I$; it is necessary only to check that this is well defined. The kernel of $\mathrm{id}_P \otimes \varepsilon^+ \colon P \otimes P \to P$ is $P \otimes P_+$, and so by the counit condition, we have $\Delta^+ p - p \otimes 1 \in P \otimes P_+$ for all $p \in P$. For any $i \in I$, we have $p(r + i) - p(r) = (\Delta^+ p - p \otimes 1)(r, i)$. By 5.4, the right-hand side of this equality is in I, and so the action is well defined.

(4) $\Rightarrow$ (3): Consider the set J of elements of I that are sent into I by all elements of P_+. If $f \in P_+$, then $\Delta^+ f \in P_+ \otimes P + P \otimes P_+$. Thus for $j, k \in J$, Lemma 5.4 implies $f(j + k) \in I$ and hence $j + k \in J$. Similarly, $\Delta^\times f \in P_+ \otimes P_+ \subset P \otimes P_+$, and so for $r \in R$ and $j \in J$, we have $f(rj) \in I$ and hence $rj \in J$. Therefore J is an ideal, and if a generating set for I is sent by P_+ into I, we have $I = J$. So all of I is preserved by P_+. $\square$

5.7. Proposition. *Let I and J be P-ideals in a P-ring A. Then IJ is a P-ideal.*

Proof. It is sufficient to check $f(xy) \in IJ$ for all $f \in P_+$, $x \in I$, and $y \in J$ because such xy form a generating set. We can write $\Delta^\times f = \sum f_i^{[1]} \otimes f_i^{[2]}$ with $f_i^{[1]}, f_i^{[1]} \in P_+$, and so we have $f(xy) = \sum f_i^{[1]}(x) f_i^{[2]}(y) \in IJ$. $\square$

6. Two-sided ideals

Let P be a k-plethory, and let P' be a k'-plethory.

6.1. An ideal J in a k-k'-biring S is called a *k-k'-ideal* if the quotient k-ring S/J admits the structure of a k-k'-biring. This is clearly equivalent to S/J being, in the opposite of Ring_k, a sub-k'-ring object of S, and so if S/J admits such a structure, it is unique. This is also equivalent to the existence of a generating set X of J such that, in the notation of 1.1, we have

1. $\Delta_S^+(X) \subseteq S \otimes J + J \otimes S$,
2. $\Delta_S^\times(X) \subseteq S \otimes J + J \otimes S$, and
3. $\beta_S(c)(X) = 0$ for all $c \in k'$.

6.2. A k-k'-ideal J in a P-P'-biring S is called a *P-P'-ideal* if there exists a P-P'-biring structure on the quotient k-k'-biring S/J such that $S \to S/J$ is a map of P-P'-birings. If such an action exists, it is unique, and so as was the case for P-ideals, being a P-P'-ideal is a property, rather than a structure.

6.3. Proposition. *Let J be a k-k'-ideal in a k-k'-biring S. Then the following are equivalent*:

1. *J is a P-P'-ideal*;
2. *J is the kernel of a map of P-P'-birings*;
3. *$P_+ \circ J \circ P' \subseteq J$*;
4. *J is generated by a set X such that $P_+ \circ X \circ P' \subseteq J$.*

The asymmetry in (3) is due to the traditional definition of ideal. If we took a more categorical approach and considered, instead of kernels of maps $R \to S$ of k-rings, the fiber products $R \times_S k$, the P_+ in (3) would become a P.

Proof. As in 5.2, the only implication that requires proof is (4) $\Rightarrow$ (1).

So, assume (4). By 5.2, J is a P-ideal; and by assumption, J is a k-k'-ideal. Therefore S/J is a P-k'-biring. For all $s \in S, j \in J, f \in P'$, we have

$$(s + j) \circ f = s \circ f + j \circ f \equiv s \circ f \mod J,$$

and so the right P'-action descends to S/J. $\square$

6.4. If J is a P-P-ideal in P itself, then this proposition implies P/J is a P-plethory in the sense that the P-P-biring structure on P/J extends to a unique P-plethory structure on P/J.

6.5. Proposition. *The category* $\mathsf{BR}_{P,P'}$ *of* P-P'-*birings has all colimits, and the forgetful functor* $\mathsf{BR}_{P,P'} \to \mathsf{Ring}_P$ *preserves them.*

Proof. Given a diagram C of P-P'-birings, its colimit S in the category of P-rings has the property that for any P-ring R, the set $\mathsf{Ring}_P(S, R)$ is the limit of the sets $\mathsf{Ring}_P(T_c, R)$, where c ranges over C. Because each $\mathsf{Ring}_P(T_c, R)$ is a P'-ring and the maps are P'-equivariant, $\mathsf{Ring}_P(S, R)$ is a P'-ring. Thus, by a remark in 1.9, S has a unique P-P'-biring structure making the maps $T_c \to S$ maps of P-P'-birings, which was to be proved. $\square$

6.6. *Free plethory on a pointed biring.* The free P-plethory Q on a P-P-biring S can be constructed as in 2.1. It comes equipped with a map $P \to Q$ of k-plethories.

Now let $f: P \to S$ be a map of P-P-birings. (This is equivalent to specifying an element $s_0 \in S$ such that $p \circ s_0 = s_0 \circ p$ for all $p \in P$.) Then the free plethory on the pointed biring S is coequalizer (6.5) of the two Q-Q-biring maps $Q \odot P \odot Q \rightrightarrows Q$ induced by sending $e \odot \alpha \odot e$, on the one hand, to $\alpha \in P = S^{\odot 0}$ and, on the other, to $f(\alpha) \in S^{\odot 1}$. By 6.4, Q is a k-plethory. It is the initial object among P-plethories P' equipped with a map $S \to P'$ such that the composite $P \to S \to P'$ agrees with the structure map $P \to P'$. An action of this plethory on a k-ring R is the same as an action of P on R together with a map $S \odot R \to R$ such that $f(p) \odot r \mapsto p(r)$ for all $p \in P, r \in R$.

At this point, it is quite easy to give an explicit construction of Λ_p that does not rely on symmetric functions. Let $S = \mathbf{Z}[e, \theta_1]$ be the $\mathbf{Z}\langle e \rangle$-pointed $\mathbf{Z}$-$\mathbf{Z}$-biring determined by

$$\Delta^+ : \theta_1 \mapsto \theta_1 \otimes 1 + 1 \otimes \theta_1 - \sum_{i=1}^{p-1} \frac{1}{p} \binom{p}{i} e^i \otimes e^{p-i}, \tag{6.6.1}$$

$$\Delta^\times : \theta_1 \mapsto e^p \otimes \theta_1 + \theta_1 \otimes e^p + p\theta_1 \otimes \theta_1. \tag{6.6.2}$$

Then Cartier's Dieudonné–Dwork lemma implies Λ_p is the free $\mathbf{Z}$-plethory on S. Of course, this is just a plethystic description of Joyal's approach [10] to the p-typical Witt vectors.

6.7. The following asymmetric variant of this construction will be used in Section 7. Let P_0 be a k-plethory, let P be a P_0-plethory, let S be a P_0-P-biring, and let $g: P \to S$ be a map of P_0-P-birings. Let Q denote the free P_0-plethory on S viewed as a pointed P_0-P_0-biring. Then we have two maps of P_0-P_0-birings $S \odot_{P_0} P \rightrightarrows Q$

given by $s \odot \alpha \mapsto s \odot g(\alpha) \in S^{\odot 2}$ and $s \odot \alpha \mapsto s \circ \alpha \in S^{\odot 1}$. These then induce two maps of Q-Q-birings $Q \odot_{P_0} S \odot_{P_0} P \odot_{P_0} Q \rightrightarrows Q$. The coequalizer T of these maps is a P_0-plethory (6.4), but the two maps $P \to Q$ become equal in T, and so T is in fact a P-plethory. An action of T on a ring R is the same as an action of P on R together with a map $S \odot_P R \to R$ such that $g(\alpha) \odot r \mapsto \alpha \circ r$.

7. Amplifications over curves

Let $\mathcal{O}$ be a Dedekind domain, and let $\mathfrak{m}$ be an ideal; let k denote the residue ring $\mathcal{O}/\mathfrak{m}$, and let K denote the subring of the field of fractions of $\mathcal{O}$ consisting of elements that are integral at all maximal ideals not dividing $\mathfrak{m}$. The $\mathfrak{m}$-torsion submodule of an $\mathcal{O}$-module M is the set of $m \in M$ for which there exists an $n \in \mathbf{N}$ such that $\mathfrak{m}^n m = 0$. We say an $\mathcal{O}$-module is $\mathfrak{m}$-torsion-free if its $\mathfrak{m}$-torsion submodule is trivial, or equivalently, if it is flat locally at each maximal ideal dividing $\mathfrak{m}$.

Now let P be an $\mathcal{O}$-plethory that is $\mathfrak{m}$-torsion-free, let Q be a k-plethory, and let $f : P \to Q$ be a surjective map of plethories agreeing with the canonical map on scalars. A *P-deformation of a Q-ring* is an $\mathfrak{m}$-torsion-free P-ring R such that the action of P on $k \otimes R$ factors through an action of Q on $k \otimes R$. (Note that because $P \to Q$ is surjective, it can factor in at most one way.) A morphism of P-deformations of Q-rings is by definition a morphism of the underlying P-rings.

The purpose of this section is then to construct an $\mathcal{O}$-plethory P', *the amplification of P along Q*, such that $\mathfrak{m}$-torsion-free P'-rings are the same as P-deformations of Q-rings. It is constructed simply by adjoining $\mathfrak{m}^{-1} \otimes I$ to P, where I is the kernel of the map $P \to Q$, and so it is analogous to an affine blow-up of rings. Note however that there are some minor subtleties involved in adjoining these elements because a plethory involves co-operations, not just operations, and because we need to know how to compose elements of P with elements of $\mathfrak{m}^{-1} \otimes I$, but P may not even act on K, let alone preserve $\mathfrak{m}$.

7.1. Theorem. *The P-plethory P' of* 7.6 *is $\mathfrak{m}$-torsion-free, and the forgetful functor from the full category of $\mathfrak{m}$-torsion-free P'-rings to* Ring_P *identifies it with the category of P-ring deformations of Q-rings. Furthermore, P' has the following universal property*: *Let P'' be a P-plethory whose underlying P-ring is a P-deformation of a Q-ring. Then there is a unique map $P' \to P''$ of P-rings commuting with the maps from P, and this map is a map of P-plethories.*

7.2. Corollary. *Let P'' be a P-plethory with the property that the forgetful functor from the full category of $\mathfrak{m}$-torsion-free P''-rings to* Ring_P *identifies it with the category of P-ring deformations of Q-rings. Then there is a unique map $P'' \to P'$ of P-rings*; *this map is a map of P-plethories, and it identifies P' with the largest $\mathfrak{m}$-torsion-free P''-ring quotient of P''.*

We prove these at the end of this section. Note that either the theorem or the construction of 7.6 implies amplification is functorial in P and Q.

7.3. Remark. As always, either universal property determines P' uniquely up to unique isomorphism. The final statement of the corollary determines it without any mention of universal properties: it is the unique $\mathfrak{m}$-torsion-free P-plethory such that the forgetful functor identifies $\mathfrak{m}$-torsion-free P'-rings with P-deformations of Q-rings.

One could also describe the category of all P'-rings as the category obtained from the category of P-deformations of Q-rings (i.e., $\mathfrak{m}$-torsion-free P'-rings) by adjoining certain colimits. This would give another satisfactory approach to the functor of p-typical Witt vectors circumventing any discussion of plethories.

7.4. Lemma. *Let T be an $\mathcal{O}$-plethory. Then the T-ideal in T generated by the $\mathfrak{m}$-torsion ideal is a T-T-ideal.*

Proof. Let I denote the ideal of $\mathfrak{m}$-torsion in T, and let J denote the T-ideal it generates. First we show I is an $\mathcal{O}$-$\mathcal{O}$-ideal. Because I is $\mathfrak{m}$-torsion, the ideal $T \otimes I + I \otimes T$ is contained in the $\mathfrak{m}$-torsion ideal of $T \otimes T$. But this containment is actually an equality: because T/I is $\mathfrak{m}$-torsion-free and because $\mathcal{O}$ is a Dedekind domain, $T/I \otimes T/I$ is $\mathfrak{m}$-torsion-free. It therefore follows that $\Delta^+(I)$ and $\Delta^\times(I)$ are both contained in $T \otimes I + I \otimes T$. And last, $\beta(c)(I)$ is zero because it is torsion but $\mathcal{O}$ is torsion-free. By 6.1, the ideal I is an $\mathcal{O}$-$\mathcal{O}$-ideal.

Now we show J is a T-T-ideal. It is a T-ideal by definition, and so we need only show $J \circ T \subseteq J$, and in fact only $I \circ T \subseteq J$. So take $i \in I$ and $\alpha \in T$. Then there is some $n \in \mathbf{N}$ such that $\mathfrak{m}^n i = 0$, and for every $x \in \mathfrak{m}^n$, we have $x(i \circ \alpha) = (xi) \circ \alpha = 0$. □

7.5. *Maximal $\mathfrak{m}$-torsion-free quotient of an $\mathcal{O}$-plethory*. Let T_0 be an $\mathcal{O}$-plethory, let J denote the T_0-ideal generated by the $\mathcal{O}$-torsion. By 7.4 and 6.4, the quotient $T_1 = T/J$ is an $\mathcal{O}$-plethory. Let T_2 be the same construction applied to T_1, and so on. Then the colimit of the sequence

$$T_0 \longrightarrow T_1 \longrightarrow \cdots$$

in the category of T_0-T_0-birings (6.5) is clearly the largest $\mathfrak{m}$-torsion-free T_0-ring quotient of T_0. It is an $\mathcal{O}$-plethory because it is a quotient T_0-T_0-biring of T_0.

Note that $\mathfrak{m}$-torsion-free T_0-rings are the same as $\mathfrak{m}$-torsion-free T'-rings.

7.6. *Amplification P' of P along Q*. Let I denote the kernel of the map $P \to Q$, and let S denote the sub-$\mathcal{O}$-ring of $K \otimes P$ generated by $\mathfrak{m}^{-1} \otimes I$. (Here, all tensor products are over $\mathcal{O}$, and as usual $\mathfrak{m}^{-1}$ denotes the $\mathcal{O}$-dual of $\mathfrak{m}$ viewed as a submodule of K.) Note that we have $1 \otimes P \subseteq S$ and also that $K \otimes P$ is a $K\langle e\rangle$-P-biring, but it need not be a K-plethory.

The K-$\mathcal{O}$-biring structure on $K \otimes P$ induces an $\mathcal{O}$-$\mathcal{O}$-biring structure on S as follows: Let Δ denote either Δ^+ or $\Delta^\times$, and let Δ_K denote $\mathrm{id}_K \otimes \Delta$. Then we have

$$\begin{aligned} \Delta_K(\mathfrak{m}^{-1} \otimes I) &\subseteq \mathfrak{m}^{-1} \otimes \Delta(I) \\ &\subseteq \mathfrak{m}^{-1} \otimes (P \otimes I + I \otimes P). \end{aligned}$$

Identifying $K \otimes P \otimes P$ with $(K \otimes P) \otimes (K \otimes P)$, we have

$$\Delta_K(\mathfrak{m}^{-1} \otimes I) \subseteq (1 \otimes P) \otimes (\mathfrak{m}^{-1} \otimes I) + (\mathfrak{m}^{-1} \otimes I) \otimes (1 \otimes P) \subseteq S \otimes S.$$

Because Δ_K is an $\mathcal{O}$-ring map, it follows that $\Delta(S) \subseteq S \otimes S$. Similarly, if ε denotes either the additive or multiplicative counit and $\varepsilon_K = \mathrm{id}_K \otimes \varepsilon$, then

$$\varepsilon_K(\mathfrak{m}^{-1} \otimes I) = \mathfrak{m}^{-1} \otimes \varepsilon(I) \subseteq \mathfrak{m}^{-1} \otimes \mathfrak{m} = \mathcal{O},$$

and as above, we have $\varepsilon(S) \subseteq \mathcal{O}$. The properties necessary for this data to give a $\mathcal{O}$-$\mathcal{O}$-biring structure on S follow from the K-$\mathcal{O}$-biring properties on $K \otimes P$.

Because I is preserved by the right action of P, so is S, and therefore S has a $\mathcal{O}\langle e\rangle$-$P$-biring structure. Let T be the construction of 6.7 applied to the $\mathcal{O}$-plethory P, the $\mathcal{O}\langle e\rangle$-$P$-biring S, and the inclusion map $P \to S$.

Finally, let P' denote the maximal $\mathfrak{m}$-torsion-free quotient of T (7.5). It is a P-plethory because T is.

7.7. Lemma. *Let R be an $\mathfrak{m}$-torsion-free P-ring. Then the action of P on R factors through at most one action of P', and one exists if and only if R is a P-deformation of a Q-ring.*

Proof. Suppose the action of P on R prolongs to two actions $\circ_1$ and $\circ_2$ of P'. For any $\alpha \in P'$ and $r \in R$, we want to show $\alpha \circ_1 r = \alpha \circ_2 r$. Because T surjects onto P', it is enough to show this for α in T and, because S generates T, even in S. But S is a subset of $K \otimes P$; so take some $n \in \mathbf{N}$ such that $\mathfrak{m}^n \alpha \subseteq P$. Then

$$x(\alpha \circ_1 r) = (x\alpha) \circ_1 r = (x\alpha) \circ_2 r = x(\alpha \circ_2 r)$$

for all $x \in \mathfrak{m}^n$. But because R is $\mathfrak{m}$-torsion-free, we have $\alpha \circ_1 r = \alpha \circ_2 r$, and so there is at most one compatible action of P' on R.

The action of P on $R/\mathfrak{m}R$ factors through Q if and only if $I \circ R \subseteq \mathfrak{m}R$. This is equivalent to $(\mathfrak{m}^{-1} \otimes I) \circ R \subseteq R$ under the map

$$(K \otimes P) \odot R = K \otimes (P \odot R) \xrightarrow{\circ} K \otimes R,$$

which is in turn equivalent to $S \circ R \subset R$. Because R is $\mathfrak{m}$-torsion-free and because $K \otimes S = K \otimes P$, this is then equivalent to the existence of some map $\circ' \colon S \odot_P R \to R$

of $\mathcal{O}$-rings such that $p \circ' r = p \circ r$ for all $p \in P, r \in R$. By 6.7, this is equivalent to an action of T on R that is compatible with the given action of P, and this is equivalent to such an action of P' on R. $\square$

7.8. Proof of 7.1. P' is $\mathfrak{m}$-torsion-free by construction.

The forgetful functor is clearly faithful, and Lemma 7.7 implies its image is as stated. To see it is full, let R and R' be $\mathfrak{m}$-torsion-free P'-rings and let $f: R \to R'$ be a map of P-rings. We need to check $f(\alpha \circ r) = \alpha \circ f(r)$ for all $\alpha \in P'$ and $r \in R$. As in the proof of 7.7, it is enough to show this for α in S, where the equality follows because R' is $\mathfrak{m}$-torsion-free. This proves the functor is fully faithful.

Let P'' be as in the universal property. By the previous paragraph, the action of P on P'' extends uniquely to an action of P'; and because P' is the free P'-ring on one element, there is a unique map of P'-rings $P' \to P''$ sending e to e. Again by the previous paragraph, we see there is a unique map $P' \to P''$ of P-rings sending e to e, that is, commuting with the maps from P.

To show this is a map of P-plethories, it is enough to show there exists some map $P' \to P''$ of P-plethories. Because P'' is $\mathfrak{m}$-torsion-free, such a map is the same as a map $T \to P''$ of P-plethories, and this is the same as a map $S \to P''$ of $\mathcal{O}\langle e \rangle$-$P$-birings respecting the maps from P. Because P'' is $\mathfrak{m}$-torsion-free, there is at most one such map, and there is exactly one if the map $P \to P''$ sends I to $\mathfrak{m}P''$. But this is just another way of saying the P-ring underlying P'' is a P-deformation of a Q-ring, and that fact we are given. $\square$

7.9. Proof of 7.2. Replacing P'' with its maximal $\mathfrak{m}$-torsion-free quotient (7.5), we can assume P'' is $\mathfrak{m}$-torsion-free. Then P'' and P' are both initial objects in the category of P-deformations of Q-rings and so are uniquely isomorphic. The universal property of the theorem applied to P'' then implies this isomorphism is a map of $\mathcal{O}$-plethories. $\square$

7.10. Suppose K admits a P-action. Then K is trivially a P-deformation of a Q-ring and, by 7.1, has a unique compatible P'-action. By 1.13, there is a canonical K-plethory structure on $K \otimes P'$.

Propostion. *If K admits a P-action, the map $K \otimes P \to K \otimes P'$ is an isomorphism of K-plethories. Moreover, under this identification, P' is the $\mathcal{O}$-subring of $K \otimes P$ generated by the $\circ$-words in the elements of $\mathfrak{m}^{-1} \otimes I$.*

Proof. To show the first statement, it is enough to show the map induces an equivalence between $\mathsf{Ring}_{K \otimes P}$ and $\mathsf{Ring}_{K \otimes P'}$. But a $K \otimes P$-ring structure on a K-ring R is the same (by 1.13) as an action of P on R, and because R is trivially a P-deformation of a Q-ring, this is the same as a P'-action, which (by 1.13 again) is the same as a $K \otimes P'$-ring structure on R.

Because P' is $\mathfrak{m}$-torsion-free, it is naturally an $\mathcal{O}$-subring of $K \otimes P' = K \otimes P$. Since P' is the surjective image of the free plethory on the biring S, it is the smallest $\mathcal{O}$-ring in $K \otimes P$ containing $\mathfrak{m}^{-1} \otimes I$ (and hence S) and closed under composition. $\square$

8. The cotangent algebra

By the structure of an *algebra over* k on an $\mathbf{Z}$-algebra A, we mean simply a morphism $k \to A$ of $\mathbf{Z}$-algebras. The image need not be central. These form a category in the obvious way.

For any k-k'-biring S, write C_S for the k-module $S_+/S_+^2 = \ker(\varepsilon_S^+)/\ker(\varepsilon_S^+)^2$. It is called the cotangent space of S. The purpose of this section is to show that the cotangent space is naturally a k-k'-bimodule and, especially, the cotangent space of a k-plethory is naturally an algebra over k. We do this by showing that if S' is a k'-k''-biring, then $C_{S \odot S'} = C_S \otimes_{k'} C_{S'}$. Thus, when $k = k' = k''$, the cotangent space is a monoidal functor, so it sends plethories (monoids in the category of k-k-birings) to algebras over k (monoids in the category of k-k-bimodules).

First we show all elements of S_+ are additive up to second order:

8.1. Lemma. *Let J denote the kernel of the map $\varepsilon^+ \otimes \varepsilon^+ \colon S \otimes_k S \to k$. Then for all $s \in S_+$, we have $\Delta^+(s) \equiv s \otimes 1 + 1 \otimes s \bmod J^2$.*

Proof. The cotangent space functor takes coproducts in Ring_k to coproducts of k-modules and (hence) takes cogroup objects to cogroup objects. In particular, we have an identification $J/J^2 = C_S \oplus C_S$, and under this identification, the map $C_S \to C_S \oplus C_S$ of cotangent spaces induced by Δ^+ makes C_S a cogroup in the category of k-modules. But the only cogroup structure on a k-module is the diagonal map. $\square$

8.2. Proposition. *Consider the right action of k', as a monoid, on S given by setting $s \cdot c$ to be the image of $s \odot ce$ under the identification $S \odot_{k'} k'\langle e\rangle = S$. (Explicitly, $s \cdot c = \sum \beta(c)(s_i^{[1]})s_i^{[2]}$.) Then this action preserves S_+ and descends to C_S, and the resulting action makes the k-module C_S a k-k'-bimodule.*

Proof. The action preserves S_+ since $\varepsilon^+(s) = s \cdot 0$. Because it acts by ring endomorphisms, it also preserves S_+^2, and thus it descends to C_S. By 8.1, k' acts not just as a monoid, but as a ring. It commutes with the k-action because for any $b \in k$, we have $(bs) \odot (ce) = (b \odot (ce))(s \odot (ce)) = b(s \odot (ce))$ in $S \odot k'\langle e\rangle$. $\square$

8.3. Proposition. *The map $k \to C_{k\langle e\rangle}$ given by $c \mapsto ce$ is an isomorphism of k-k-bimodules. If S is a k-k'-biring and S' a k'-k''-biring, then the map $C_S \otimes_{k'} C_{S'} \to C_{S \odot_{k'} S'}$ given by $s \otimes s' \mapsto s \odot s'$ is well defined and an isomorphism of k-k'-bimodules.*

Proof. The first statement follows immediately from the definition (1.2) of $k\langle e\rangle = k[e]$.

Now we will show the second map is well-defined. Note that $\varepsilon^+(s \odot s') = s(s'(0))$, where $\alpha(c)$ denotes $\beta(c)(\alpha)$. Thus if $s \in S_+$ and $s' \in S'_+$, then $s \odot s' \in (S \odot S')_+$, and so we have a well-defined map $S_+ \times S'_+ \to C_{S \odot S'}$. This map is clearly additive in the first variable and is additive in the second by 8.1. Thus to check that it descends to $C_S \times C_{S'}$, we need only show $s \odot s' \in (S \odot S')_+^2$ for $s \in S_+^2$ and $s \odot s' \in (S \odot S')_+^2$ for $s' \in (S')_+^2$. The first is clear, for ring operations come out of the left side of the composition product. For the second, s' may be a sum of products, but up to second order, sums also come out of the right side (by 8.1), and so we may assume $s' = s'_1 s'_2$, $s'_i \in S'_+$. Then $s \odot s'_1 s'_2 = \Delta^\times s(s'_1, s'_2)$, but $\Delta^\times s \in S_+ \otimes S_+$ by 5.5. Elements of k' may be moved between the factors by the identifications

$$S' \odot (k'\langle e\rangle \odot S'') = S' \odot S'' = (S' \odot k'\langle e\rangle) \odot S''$$

and so the map descends to $C_S \otimes_{k'} C_{S'}$. Finally, it is a map of k-k''-bimodules by the associativity of the composition product.

Since the map $C_S \otimes_{k'} C_{S'} \to C_{S \odot_{k'} S'}$ is all we need to make the cotangent space of a plethory into an algebra, we leave the many details of the isomorphism to the reader. The key observation is that

$$s \odot s' = s \odot (e + \varepsilon^+(s')) \circ (e - \varepsilon^+(s')) \circ s' = s \circ (e + \varepsilon^+(s')) \odot (s' - \varepsilon^+(s')),$$

so that $S \odot_{k'} S'$ is generated by elements of the form $s \odot s'$ with $s' \in S'_+$. This suggests the map of rings $f : S \odot S' \to k \oplus C_S \otimes C_{S'}$ given by $f(s \odot s') = \varepsilon^+(s \odot s') + (s \circ (e + \varepsilon^+(s')) - \varepsilon^+(s \odot s')) \otimes (s' - \varepsilon^+(s'))$, which descends to the inverse $C_{S \odot S'} \to C_S \otimes C_{S'}$. □

8.4. *C_P is an algebra over k.* Let P be a k-plethory. The composition $P \odot P \to P$ and unit $k\langle e\rangle \to P$ induce $C_P \otimes_k C_P \to C_P$ and $k = C_{k\langle e\rangle} \to C_P$ making C_P an algebra over k. Note that e is the unit for composition and thus the unit of this algebra.

8.5. *I/I^2 is a C_P-module.* Let I be a P-ideal in a P-ring R. Then by 5.7, C_P acts as a monoid on I/I^2. But 8.1 implies this action is $\mathbf{Z}$-linear, and we always have $(\alpha + \beta) \circ x = \alpha \circ x + \beta \circ x$; so, this action is actually a C_P-module structure on I/I^2. The two k-module structures on I/I^2, one by way of $k \to C_P$ and the other $k \to R$, agree.

9. Twisted bialgebras and their coactions

First we recall some basic notions introduced by Sweedler [17], as modified by Takeuchi [18, 4.1].

9.1. If A and B are two algebras over k, then $A \otimes_k B$, where the k-module structure on each factor is given by multiplication on the left, has two remaining k-actions: one by right multiplication on A and one by right multiplication on B. Let $A \circledast B$, the *Sweedler product*, denote the subgroup where these two actions coincide. It is an algebra over k

with multiplication

$$\left(\sum_i a_i \otimes b_i\right)\left(\sum_j a'_j \otimes b'_j\right) = \sum_{i,j} a_i a'_j \otimes b_i b'_j.$$

The Sweedler product is symmetric in the sense that the symmetrizing map

$$A \otimes_k B \to B \otimes_k A, \quad a \otimes b \to b \otimes a$$

sends $A \circledast B$ isomorphically to $B \circledast A$. Note that $\circledast$ is not naturally associative in the generality above (but it is if, say, the algebras are k-flat [17, Section 2]).

If M and N are left A-modules, then $M \otimes_k N$ is a left $A \circledast A$-module by $(\sum_i a_i \otimes b_i)(m \otimes n) = \sum_i a_i m \otimes b_i n$.

9.2. We say A is a *twisted k-bialgebra* if it is equipped with a map $\Delta\colon A \to A \circledast A$ of algebras over k and a map $\varepsilon\colon A \to k$ of k-modules satisfying the following properties

1. the composite $A \xrightarrow{\Delta} A \circledast A \hookrightarrow A \otimes A$ is coassociative with counit ε, and
2. $\varepsilon(1) = 1$ and for all $a, b \in A$, we have $\varepsilon(ab) = \varepsilon(a\iota(\varepsilon(b)))$, where ι denotes the structure map $k \to A$.

Thus, the structure of a twisted k-bialgebra on A is the same as the structure of a k-bialgebroid on A where the structure map $k \otimes_{\mathbf{Z}} k \to A$ factors through multiplication $k \otimes_{\mathbf{Z}} k \to k$. (Several equivalent formulations of the notion of bialgebroid are discussed in Brzezinski–Militaru [3].) Assuming flatness, it is also the same as what Sweedler [17] called a $\times_k$-bialgebra structure.

The category of left A-modules then has a monoidal structure that is compatible with $\otimes_k$, and this is precisely the data needed to make this so ([16, 5.1], [3, 3.1]). If Δ is cocommutative in the obvious sense, this monoidal category is symmetric.

9.3. Let C be an algebra over k. A coaction of A on C is a map $\alpha\colon C \to A \circledast C$ of algebras commuting with the maps from k such that the composite

$$C \xrightarrow{\alpha} A \circledast C \hookrightarrow A \otimes C$$

is a coaction of A, viewed as a k-coalgebra, on C. (So, C is a left A-comodule algebra in the terminology of [5]). Given a left A-module M, a left C-module N, and a coaction of A on C, the tensor product $M \otimes_k N$ is naturally a left C-module by way of α. In this way, the category of left A-modules acts on the category of left C-modules.

The map $\Delta\colon A \to A \circledast A$ is a coaction, the *regular* coaction.

9.4. *Generalized semi-direct product* $R \rtimes_A C$. Suppose A coacts on C and also acts on a k-ring R in the sense that the multiplication map $R \otimes R \to R$ is a map of A-modules. Then $R \otimes_k C$ is an R-module and (by 9.3) a C-module, and this induces a multiplication

$$(R \otimes C) \otimes (R \otimes C) = R \otimes (C \otimes (R \otimes C)) \longrightarrow R \otimes (R \otimes C) \longrightarrow R \otimes C$$

on $R \otimes C$ with unit $1 \otimes 1$. The map $k \to R \otimes C$ is simply $x \mapsto x \otimes 1 = 1 \otimes x$.

We denote this algebra by $R \rtimes_A C$. When C is A with the regular coaction and A is untwisted (i.e., the image of k is in the center of A), this agrees with the semi-direct, or "smash", product in the usual sense [5].

It is immediate that the map $R \to R \rtimes_A C$ given by $r \mapsto r \otimes 1$ is a map of algebras over k, and the counit property implies the map $C \to R \rtimes_A C$, $c \mapsto 1 \otimes c$ is also such a map. Therefore an $R \rtimes_A C$-module structure on a k-module M is the same as actions of R and C on M which are intertwined as follows:

$$c(r(c'm)) = \sum_i (c_i^{(1)} r)(c_i^{(2)} c'm),$$

where $\Delta(c) = \sum_i c_i^{(1)} \otimes c_i^{(2)} \in A \otimes C$.

10. The additive bialgebra

The purpose of this section is to show that the set of additive elements in a k-plethory is naturally a cocommutative twisted k-bialgebra, at least under certain flatness hypotheses.

10.1. Let P be a k-plethory. An element $f \in P$ is *additive* if $\Delta^+(f) = f \otimes 1 + 1 \otimes f$, which is equivalent to requiring that $f(x+y) = f(x) + f(y)$ for all elements x, y in all P-rings. (In fact, taking $x = e \otimes 1$, $y = 1 \otimes e$ in $P \otimes P$ suffices.) Because we have $\varepsilon^+(f) = f(0) = 0$, every additive element is in P_+. The set A, or A_P, of additive elements is clearly closed under addition and composition, and composition by additive elements distributes over addition; thus A is a generally non-commutative algebra with unit $1_A = e$. Furthermore, the map $\iota : k \to A, c \mapsto ce$ is a map of algebras; so in this way, A is an algebra over k.

10.2. Proposition. *The image of $A \circledast A$ in $P \otimes P$ is the set of k-interlinear elements, where $f \in P \otimes P$ is said to be k-interlinear if $f(r,s)$ is additive in each argument $r, s \in R$ and we have $f(cr, s) = f(r, cs)$ for all $c \in k$.*

Here we are using the notation of 5.3. Note that a k-interlinear element f is not required to be k-linear in each argument.

Proof. First we show that the image of $A \otimes P$ is the set of elements that are additive on the left. If f is in the image of $A \otimes P$, it is immediate that f is additive on the left. Now suppose f is additive on the left. Because A is the kernel of the

k-module map

$$P \xrightarrow{\varphi} P \otimes P, \quad f \mapsto \Delta^+(f) - f \otimes 1 - 1 \otimes f, \tag{10.2.1}$$

the image of $A \otimes P$ in $P \otimes P$ is the kernel of the map $\varphi \otimes 1 \colon P \otimes P \to P \otimes P \otimes P$; so it is enough to show f is in the kernel of $\varphi \otimes 1$. Write $f = \sum_i \alpha_i \otimes \beta_i$. Then we have

$$\begin{aligned}
\sum_i \Delta^+(\alpha_i) \otimes \beta_i &= \sum_i \alpha_i(e \otimes 1 + 1 \otimes e) \otimes \beta_i \\
&= \sum_i (\alpha_i \otimes \beta_i) \circ (e \otimes 1 \otimes 1 + 1 \otimes e \otimes 1, 1 \otimes 1 \otimes e) \\
&= f(e \otimes 1 \otimes 1 + 1 \otimes e \otimes 1, 1 \otimes 1 \otimes e) \\
&= f(e \otimes 1 \otimes 1, 1 \otimes 1 \otimes e) + f(1 \otimes e \otimes 1, 1 \otimes 1 \otimes e) \\
&= \sum_i \alpha_i(e) \otimes 1 \otimes \beta_i(e) + 1 \otimes \alpha_i(e) \otimes \beta_i(e) \\
&= \sum_i (\alpha_i \otimes 1 + 1 \otimes \alpha_i) \otimes \beta_i.
\end{aligned}$$

But $(\varphi \otimes 1)(f)$ is the difference between the first and last sums, and so f is in the kernel.

Essentially the same argument shows the image of $A \otimes A$ in $A \otimes P$ is the set of elements whose image in $P \otimes P$ is both left additive and right additive.

It is clear that any element in the image of $A \circledast A$ is interlinear. Now let f be a k-interlinear element of $A \otimes A$. Then $f(ce \otimes 1, 1 \otimes e) = f(e \otimes 1, 1 \otimes ce)$. Writing $f = \sum_i \alpha_i \otimes \beta_i$, we have

$$\sum_i (\alpha_i \circ (ce)) \otimes \beta_i = \alpha_i \otimes (\beta_i \circ (ce)),$$

that is, f transforms the same way under the two actions of k on A by right multiplication. $\square$

10.3. Proposition. *$\Delta^\times(A)$ is contained in the image of $A \circledast A$ in $P \otimes P$. If the maps $A^{\otimes 2} \to P^{\otimes 2}$ and $A^{\otimes 3} \to P^{\otimes 3}$ are injective, the algebra A is a cocommutative twisted k-bialgebra* (9.2)*, where ε is $\varepsilon^\times$ and Δ is $\Delta^\times$, viewed as a map $A \to A \circledast A \subseteq A \otimes A$.*

Proof. For any element $f \in A$ and any P-ring R, the map $R \times R \to R$ given by $(r, s) \mapsto f(rs)$ is clearly k-interlinear. Because this map is just the application of $\Delta^\times(f)$, we see $\Delta^\times(f)$ is k-interlinear and therefore lies in the image of $A \circledast A$, by 10.2.

Now we show Δ is a map of algebras over k. Take $a, b \in A$. Because a is additive and using 1.12, we have

$$\Delta^{\times}(a \circ b) = a \circ \Delta^{\times}(b) = \sum_{i,j} (a_i^{[1]} \circ b_j^{[1]}) \otimes (a_i^{[2]} \circ b_j^{[2]}),$$

but this last term is the product in $A \circledast A$ of $\Delta^{\times}(a)$ and $\Delta^{\times}(b)$. It is clear that Δ is a map over k.

The cocommutativity of Δ follows from that of $\Delta^{\times}$.

It remains to check properties (1)–(2) of 9.2. Because we have $A \otimes A \otimes A \subseteq P \otimes P \otimes P$, the coassociativity of Δ can be tested in $P \otimes P \otimes P$, where it follows from the associativity of the comultiplication $\Delta^{\times}$ on P. The map ε is a counit for Δ simply because $\varepsilon^{\times}$ is for $\Delta^{\times}$.

It is clear that $\varepsilon(1) = 1$. By 1.12, we also have (ι denoting the structure map $k \to A$)

$$\varepsilon^{\times}(a \circ \iota(\varepsilon^{\times}(b))) = a \circ \varepsilon^{\times}(e\varepsilon^{\times}(b)) = a \circ ((e\varepsilon^{\times}(b))(1)) = a \circ \varepsilon^{\times}(b) = \varepsilon^{\times}(a \circ b),$$

for all $a, b \in A$. $\square$

10.4. Remark. If A and P are flat over k, then the injectivity hypotheses of the proposition hold. In particular, they do if k is a Dedekind domain and P is torsion-free. They also hold if the inclusion $A \to P$ is split, for example if $P = S(A)$.

In fact, we do not know any examples of plethories where the assumptions of the previous proposition are not satisfied, but if they exist, it seems clear that the correct replacement of A would be the collection of all multilinear elements in all tensor powers of P assembled together in some sort of operadic coalgebra construction.

11. The coaction of A_P on C_P

Because $A = A_P$ is contained in P_+, we have a map $A \to C_P$, which is clearly a map of algebras over k.

11.1. Proposition. *There is a unique map v such that the diagram*

$$\begin{tikzcd} P_+ \arrow[r, "\Delta^{\times}"] \arrow[d] & P_+ \otimes P_+ \arrow[d] \\ C_P \arrow[r, dashed, "v"] & P_+ \otimes C_P \end{tikzcd}$$

(*using* 5.5) *commutes*; *and the image of v is contained in the image of $A \circledast C_P$.*

If the maps $A^{\otimes i} \otimes C_P \to P_+^{\otimes i} \otimes C_P$ are injective for $i = 1, 2$, then v, viewed as a map $C_P \to A \circledast C_P$, is a coaction of the twisted k-bialgebra A on C_P, and the natural map $A \to C_P$ is A-coequivariant, where A has the regular coaction.

The injectivity hypotheses hold under the flatness and splitting hypotheses of 10.4.

Proof. The first statement is immediate because $\Delta^{\times}: P \to P \otimes P$ is a ring map.

Let φ be as in (10.2.1). To show the image of v is contained in the image of $A \otimes C_P$, it is enough to show the composite map along the bottom row of the diagram

$$\begin{tikzcd} P_+ \arrow[r, "\Delta^{\times}"] \arrow[d] & P_+ \otimes P_+ \arrow[r, "\varphi\otimes 1"] \arrow[d] & P_+ \otimes P_+ \otimes P_+ \arrow[d] \\ C_P \arrow[r, "v"] & P_+ \otimes C_P \arrow[r, "\varphi\otimes 1"] & P_+ \otimes P_+ \otimes C_P \end{tikzcd}$$

is zero, and hence it is enough to show the composite of the maps along the top and the right is zero. The method is the same as that of 10.2.

For any $f \in P_+$, write $\Delta^{\times}(f) = \sum_i f_i^{[1]} \otimes f_i^{[2]}$. Then

$$\sum_i \Delta^+(f_i^{[1]}) \otimes f_i^{[2]} = \sum_i f_i^{[1]}(e \otimes 1 + 1 \otimes e) \otimes f_i^{[2]}(e) = f((e \otimes 1 + 1 \otimes e) \otimes e).$$

On the other hand, by 8.1 we can write $\Delta^+(f) \equiv f \otimes 1 + 1 \otimes f \bmod J^2$, where $J = P \otimes P_+ + P_+ \otimes P$. Therefore, we have

$$\begin{aligned} f(e \otimes 1 \otimes e + 1 \otimes e \otimes e) &\equiv f(e \otimes 1 \otimes e) + f(1 \otimes e \otimes e) \bmod P \otimes P \otimes P_+^2 \\ &= \sum_i (f_i^{[1]}(e) \otimes 1 + 1 \otimes f_i^{[1]}(e)) \otimes f_i^{[2]} \end{aligned}$$

and hence

$$\begin{aligned} (\varphi \otimes 1)(\Delta^{\times}(f)) &= \sum_i (\Delta^+(f_i^{[1]}) - f_i^{[1]}(e) \otimes 1 + 1 \otimes f_i^{[1]}(e)) \otimes f_i^{[2]} \\ &\equiv 0 \bmod P \otimes P \otimes P_+^2, \end{aligned}$$

which was to be proved.

As in 10.2, we show $v(f)$ is contained in the image of $A \circledast C_P$ by applying f to the equation $ce \otimes e = e \otimes ce$, for any $c \in k$.

Now we show v is a map of algebras over k. Suppose $f, g \in C_P$, and write $v(f) = \sum_i f_i^{[1]} \otimes f_i^{[2]}$ and $v(g) = \sum_j g_j^{[1]} \otimes g_j^{[2]}$ with $f_i^{[1]}, g_j^{[1]} \in A$ and $f_i^{[2]}, g_j^{[2]} \in C_P$.

Then

$$\begin{aligned} v(f \circ g) &= f\left(\sum_j g_j^{[1]} \otimes g_j^{[2]}\right) \quad \text{by 1.12} \\ &= \sum_j f(g_j^{[1]} \otimes g_j^{[2]}) \\ &= \sum_{i,j} (f_i^{[1]} \circ g_j^{[1]}) \otimes (f_i^{[2]} \circ g_j^{[2]}) \\ &= \left(\sum_i f_i^{[1]} \otimes f_i^{[2]}\right) \circ \left(\sum_j g_j^{[1]} \otimes g_j^{[2]}\right) \\ &= v(f)v(g). \end{aligned}$$

And v is a map over k because $\Delta^{\times}(ce) = c(e \otimes e)$.

All that remains is to show that ε is a counit and that v is coassociative. The first follows immediately from the counit property of $\varepsilon^{\times}$, and because of our assumptions, coassociativity can be tested in $A \otimes C_P \otimes C_P$, where it follows from the fact that $\Delta^{\times}$ is coassociative on P. □

11.2. Example. If B is a cocommutative k-bialgebra and $P = S(B)$, then $C_P = B$. The image of inclusion $C_P = B \hookrightarrow S(B)$ is contained in A, and this is a section of the natural map $A \to C_P = B$. The coaction of A on B is given by this inclusion:

$$B \xrightarrow{\Delta} B \otimes B \longrightarrow A \otimes B.$$

If k is a $\mathbf{Q}$-ring, the inclusion $B \hookrightarrow A$ is an isomorphism, but if k is an $\mathbf{F}_p$-ring for some prime number p, it will never be. For we have $e^p \in A$, but the image of e^p in C_P is zero because $p \geqslant 2$.

11.3. *I/I^2 is an $R/I \rtimes_A C_P$-module*. Let I be a P-ideal in a P-ring R. Then by 8.4, I/I^2 is naturally a C_P-module. It follows from the associativity of the action of P on R that the C_P-action and R/I-action are intertwined as in 9.4, and therefore these actions extend to an action of $R/I \rtimes_A C_P$.

11.4. *$\Omega^1_{R/k}$ is an $R \rtimes_A C_P$-module*. Let R be a P-ring. Because we have $\Omega^1_{R/k} = I/I^2$, where I is the kernel of the multiplication map $R \otimes R \to R$, the R-module $\Omega^1_{R/k}$ is naturally a $R \rtimes_A C_P$-module.

12. Classical plethories revisited

Let p be a prime number. In this section we present a construction of Λ_p (of 2.13), and hence an approach to the p-typical Witt vectors, which given the generalities developed earlier in this paper, is completely effortless. We also discuss the linearization of Λ_p and similar classical plethories.

12.1. Consider the trivial $\mathbf{F}_p$-plethory $\mathbf{F}_p\langle e\rangle$. The bialgebra A of additive elements of $\mathbf{F}_p\langle e\rangle$ (see 10.3) is the free bialgebra $\mathbf{F}_p[F]$ on the monoid $\mathbf{N}$ generated by the Frobenius element $F = e^p$. It therefore has a canonical lift $\mathbf{Z}[F]$ to a commutative bialgebra over $\mathbf{Z}$. Let $\mathbf{Z}\langle F\rangle$ denote $S(\mathbf{Z}[F]) = \mathbf{Z}[F^{\circ\mathbf{N}}]$, the free $\mathbf{Z}$-plethory on this bialgebra. The natural map $\mathbf{Z}\langle F\rangle \to \mathbf{F}_p\langle e\rangle$ is a surjection, and so we can consider the amplification of $\mathbf{Z}\langle F\rangle$ along $\mathbf{F}_p\langle e\rangle$.

12.2. Proposition. *There is a unique map of $\mathbf{Z}\langle F\rangle$-rings from Λ_p to the amplification of $\mathbf{Z}\langle F\rangle$ along $\mathbf{F}_p\langle e\rangle$, and this map is an isomorphism of $\mathbf{Z}\langle F\rangle$-plethories.*

Proof. Let P' denote the amplification. Because Λ_p is p-torsion-free, 7.2 implies we need only show that a $\mathbf{Z}\langle F\rangle$-deformation of a $\mathbf{F}_p\langle e\rangle$-ring is the same as a p-torsion-free Λ_p-ring. But this is just 3.4, the strengthened form of Cartier's Dieudonné–Dwork lemma. $\square$

12.3. The same process gives ramified and twisted versions of the Witt ring. Let $\mathcal{O}$ be a Dedekind domain, let k be a residue field of characteristic p, let q be a power of p, and let F be a lift to $\mathcal{O}$ of the endomorphism $x \mapsto x^q$ of k. Then the $\mathbf{Z}$-plethory $\mathbf{Z}\langle F\rangle$ acts on $\mathcal{O}$, and we can form the plethory $\mathcal{O}\langle F\rangle := \mathcal{O} \otimes \mathbf{Z}\langle F\rangle$, which maps to $k\langle e\rangle$ by $F \mapsto e^q$. Let M denote the rank-one $\mathcal{O}$-module $\mathfrak{m}^{-1}(F - e^q)$, and let B denote $\mathcal{O}\langle e\rangle \otimes S_{\mathcal{O}}(M)$. One can easily check there is a unique $\mathcal{O}$-$\mathcal{O}$-biring structure on B such that the inclusion $B \to K\langle F\rangle$ is a map of birings. (The structure maps are similar to those in (6.6.1).) Let P denote the free pointed $\mathcal{O}$-plethory on B. Then an action of P on an $\mathfrak{m}$-torsion-free $\mathcal{O}$-ring R is the same (6.6) as a map $B \odot R \to R$ such that $e \odot r \mapsto r$, which is the same as an endomorphism $F\colon R \to R$ extending the F on $\mathcal{O}$ such that $F(x) \equiv x^q \bmod \mathfrak{m}$ for all $x \in R$. Thus an $\mathcal{O}\langle F\rangle$-deformation of an $k\langle e\rangle$-ring is the same as a P-action on an $\mathfrak{m}$-torsion-free $\mathcal{O}$-ring. Because P is $\mathfrak{m}$-torsion-free, 7.2 gives a canonical isomorphism from P to the amplification of $\mathcal{O}\langle F\rangle$ along $k\langle e\rangle$.

When $\mathfrak{m}$ is a principal ideal, surely much of this theory agrees with Hazewinkel's formula-based approach [8, Chapter 25] to objects of the same name. Any precise results along these lines would require some proficiency in his theory, which proficiency we do not have.

It seems worth mentioning, however, that when $\mathfrak{m}$ is not principal, it is unlikely $W_P(R)$ has a description in terms of traditional-looking Witt components. The reason is simply that the analogue of $W_2(R)$, the ring of length-two Λ_p-Witt vectors with entries in R, is

$$\mathsf{Ring}_{\mathcal{O}}(B, R) = \mathsf{Ring}_{\mathcal{O}}(\mathcal{O}\langle e\rangle \otimes S_{\mathcal{O}}(M), R) = R \times (\mathfrak{m} \otimes_{\mathcal{O}} R),$$

which is not naturally $R \times R$ (as sets).

12.4. It is also possible to recover Λ in this manner. For a finite set S of prime numbers, construct a $\mathbf{Z}$-plethory Φ_S as follows: Let $\Phi_{\{\}}$ denote the $\mathbf{Z}$-plethory $\mathbf{Z}\langle \psi_p \mid p \text{ prime}\rangle$, where the ψ_p are ring-like (2.7) and commute with each other. For $S' = S \cup \{p\}$, where p is a prime not in S, let $\Phi_{S'}$ denote the amplification of Φ_S along $(\mathbf{F}_p \otimes \Phi_S)/(\psi_p^{\circ n} - e^{p^n} \mid n \geqslant 0)$. Using induction, one can construct a natural map $\Phi_S \to \Lambda$ and prove that Φ_S is torsion-free and that torsion-free Φ_S-rings are the same as torsion-free $\Phi_{\{\}}$-rings such that $\psi_p(x) \equiv x^p \bmod p$ for all $p \in S$. It is also possible to show that Φ_S is canonically independent of the order of the amplifications.

Using Wilkerson's result [20] that a torsion-free λ-ring is the same a ring equipped with commuting Adams operations ψ_p such that $\psi_p(x) \equiv x^p \bmod p$ for all primes p, it follows that the maps $\Phi_S \to \Lambda$ induce an isomorphism from the colimit of the Φ_S to Λ.

One could certainly construct variants for rings of integers in general number fields, as in the single-prime case above.

12.5. *Linearization of* Λ_p. The additive bialgebra of Λ_p is $\mathbf{Z}[F]$ with comultiplication $F \mapsto F \otimes F$. (Because Λ_p is torsion-free, additivity can be checked in $\mathbf{Q} \otimes \Lambda_p = \mathbf{Q}\langle F\rangle$, to which 11.2 can be applied.) It follows—either from the traditional, explicit description (2.13) of Λ_p or from 6.6—that the cotangent space C_{Λ_p} is freely generated by the image θ of θ_1, the coaction is given by $\theta \mapsto F \otimes \theta$, and the map $\mathbf{Z}[F] \to \mathbf{Z}[\theta]$ is $F \mapsto p\theta$. Note that

$$\theta^n = p^{-n} F^n \equiv \theta_n \bmod (\Lambda_p)_+,$$

that is, the two familiar generating sets $\{\theta_n\}$ and $\{\theta_1^{\circ n}\}$ of Λ_p agree in C_{Λ_p}. Also note that the map $F \mapsto \theta$ is an isomorphism from A to C_{Λ_p} of algebras with an A-coaction, but the canonical map is not this map, or even an isomorphism at all. The general case of 12.3 is very similar, but there is no canonical element θ, only $\mathfrak{m}^{-1}F$.

12.6. *Linearization of* Λ. The situation for Λ is essentially the same. Its additive bialgebra is $\mathbf{Z}[\psi_p \mid p \text{ prime}]$ with $\Delta\colon \psi_p \mapsto \psi_p \otimes \psi_p$. The cotangent space is $\mathbf{Z}[\lambda_p \mid p \text{ prime}]$, and the coaction of $\mathbf{Z}[\psi_p \mid p \text{ prime}]$ on $\mathbf{Z}[\lambda_p \mid p \text{ prime}]$ is given by $\lambda_p \mapsto \psi_p \otimes \lambda_p$. The map $\mathbf{Z}[\psi_p \mid p \text{ prime}] \to \mathbf{Z}[\lambda_p \mid p \text{ prime}]$ is given by $\psi_p \mapsto (-1)^p p \lambda_p$. These can be checked using Newton's formulas [15, I(2.11)′].

12.7. The binomial plethory is $\Lambda/(\psi_n - e \mid n \geqslant 1)$; its additive bialgebra is the trivial one, $\mathbf{Z}$, and its cotangent algebra is $\mathbf{Q}$.

12.8. *Bloch's Frobenius.* There is an endomorphism of the de Rham–Witt complex [9], which is usually called Frobenius, but which on i-forms is $p^{-i}F$, where F is the actual Frobenius map. In fact, this endomorphism lifts to the de Rham complex of $W(R)$: By 11.4, the element $\theta \in C_P$ acts on $\Omega^1_{W(R)}$, but we have $\theta = p^{-1}F \in C_P$, and so θ reduces to Bloch's Frobenius map in degree 1. In degree $i > 0$, Bloch's Frobenius is $\theta^{\otimes i}$ as in

$$\theta^{\otimes i}(\eta_1 \wedge \cdots \wedge \eta_i) = \theta(\eta_1) \wedge \cdots \wedge \theta(\eta_i).$$

We remarked above that there is an isomorphism of A_{Λ_p} and C_{Λ_p} of algebras with an A_{Λ_p}-coaction identifying F and θ but that this is not the canonical map. This is perhaps a pleasant explanation of the meaning of the well-known fact that Bloch's is *a* Frobenius operator even though it is not *the* Frobenius operator.

For the variant of Λ_p over a general integer ring $\mathcal{O}$ at a prime $\mathfrak{m}$, the compatibility between any generalization of Bloch's Frobenius map and the true one would involve some choice of uniformizer, and so it would be a mistake to try to find such a generalization. Instead it is the $\mathcal{O}$-line $\mathfrak{m}^{-i}F^{\otimes i} = (\mathfrak{m}^{-1}F)^{\otimes i}$ that acts.

12.9. Remark. The perfect closure $(\mathbf{F}_p\langle e\rangle)^{p^{-\infty}}$ of the ring $\mathbf{F}_p\langle e\rangle$ has a unique $\mathbf{F}_p$-plethory structure compatible with that of $\mathbf{F}_p\langle e\rangle$. Let $\mathbf{Z}\langle F^{\circ\pm 1}\rangle$ denote the free $\mathbf{Z}$-plethory on the group bialgebra $\mathbf{Z}[F, F^{-1}]$ of $\mathbf{Z}$. Then the map of plethories $\mathbf{Z}\langle F^{\circ\pm 1}\rangle \to (\mathbf{F}_p\langle e\rangle)^{p^{-\infty}}$ is a surjection. One can show the amplification P of this map is the plethory push-out, or amalgamated product, of Λ_p and $\mathbf{Z}\langle F^{\circ\pm 1}\rangle$ over $\mathbf{Z}\langle F\rangle$. Its Witt functor is particularly interesting and useful: if V is an $\mathbf{F}_p$-ring, $W_P(V)$ is $A_{\mathrm{inf}}(V/\mathbf{Z}_p)$, the universal p-adic formal pro-infinitesimal $\mathbf{Z}_p$-thickening of V, in the sense of Fontaine [7, 1.2].

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