Witt vectors, lambda-rings, and arithmetic jet spaces. Copenhagen, 2016.

Exercises for week 7.

(1) Let $i\colon C' \to C$ be a fully faithful functor, and let W' be a functor $C' \to C'$. Suppose that the left Kan extension of $i \circ W'$ along i exists. Denote it by $W\colon C \to C$. Show that $i \circ W'$ is canonically isomorphic to $W \circ i$. Show that if W' has the structure of a comonad, then W also has a canonical comonad structure compatible with that on W. (Also, make the words 'canonical' and 'compatible' here precise.)

(2) Here's an example showing that $\mathfrak{p}$-torsion free modules are not closed under tensor product. Let A denote the ring $k[x,y]$, where k is a field, and let $\mathfrak{p} = (x) = xA$. Let M denote the ideal $(x,y) \subseteq A$. Prove that M is $\mathfrak{p}$-torsion free but $M \otimes_A M$ is not.

(3) Let $(A, \mathfrak{p}, F)$ be a 'context' in the sense of week 6. For any $\mathfrak{p}$-torsion free A-algebra R, let

$$W_1(R) = \{\langle a_0, a_1\rangle \in R \times R \mid a_1 \equiv F(a_0) \bmod \mathfrak{p}R\}.$$

(For example, in the usual p-typical context, this is the usual W_1.) Show that W_1 is represented by an A-A-biring Λ_1.

Write $\Psi = A[e, \psi, \psi^{\circ 2}, \dots]$ as usual. Consider the map

$$\Psi \odot \Lambda_1^{\odot m} = \Psi \odot \Lambda_1 \odot \cdots \odot \Lambda_1 \longrightarrow A[\mathfrak{p}^{-1}] \otimes_A \Psi$$

defined by $f \odot g_1 \odot \cdots \odot \cdots g_m \mapsto f \circ g_1 \circ \cdots \circ g_m$. Show that $\Lambda^{(m)}$ is the image of this map.

(4) Here is an alternative approach to the one in week 6. Consider $A[\mathfrak{p}^{-1}] \otimes_A \Psi$ again. It is a pre-Λ-ring because the Frobenius lift condition is vacuous. Show it has a minimal sub-pre-Λ-ring $\tilde{\Lambda}^{(\infty)}$ containing Ψ. You can construct it explicitly as follows. Let $\tilde{\Lambda}^{(0)} = \Psi$, and let $\tilde{\Lambda}^{(m+1)}$ be the sub-$\tilde{\Lambda}^{(m)}$-algebra of $A[\mathfrak{p}^{-1}] \otimes_A \Psi$ generated by the set $\{\delta_\eta(f) \mid \eta \in \mathfrak{p}^{-1}, f \in \tilde{\Lambda}^{(m)}\}$. Then show that if we define $\tilde{\Lambda}^{(\infty)} = \bigcup_m \tilde{\Lambda}^{(m)}$, it has the required minimality property.

Show that $\tilde{\Lambda}^{(\infty)}$ is the free $\mathfrak{p}$-torsion free pre-Λ-ring on one generator.

(5) In fact, $\tilde{\Lambda}^{(m)}$ and $\Lambda^{(m)}$ agree. So the two approaches differ only in the way the objects are constructed, not in the objects themselves. To show this, first show that $\tilde{\Lambda}^{(m)}$ is the image of the map

$$\Lambda_1^{\odot m} \odot \Psi \longrightarrow A[\mathfrak{p}^{-1}] \otimes_A \Psi$$

defined by $g_1 \odot \cdots \odot \cdots g_m \odot f \mapsto g_1 \circ \cdots \circ g_m \circ f$. Then show that there is an isomorphism

$$\Lambda_1^{\odot m} \odot \Psi \longrightarrow \Psi \odot \Lambda_1^{\odot m}$$

defined by $g_1 \odot \cdots \odot \cdots g_m \odot f \mapsto f \odot g_1 \odot \cdots \odot \cdots g_m$. Finally, invoke question 3 above.