Witt vectors, lambda-rings, and arithmetic jet spaces. Copenhagen, 2016.

Exercises for week 6. Let A, $\mathfrak{p}$, F, $W^{(m)}$, etc. be as in lecture.

(1) Let $(A, \mathfrak{p}, F)$ be the usual p-typical context $(\mathbb{Z}, p\mathbb{Z}, e^p)$. Show that $W^{(m)}(\mathbb{Z}) = \{\langle a_0, a_1, \dots \rangle \in \mathbb{Z}^{\mathbb{N}} \mid a_{n+1} \equiv a_n \bmod p^{\min(n+1,m)}\}$.

(2) Continue with the p-typical context $(\mathbb{Z}, p\mathbb{Z}, e^p)$.

Show that there is an isomorphism

$$\Lambda_p \odot \mathbb{Z}[\varepsilon]/(\varepsilon^2) \longrightarrow \mathbb{Z}[t_0, t_1, \dots]/(\dots, p^n t_n^2, \dots)$$

sending $\theta_n \odot \varepsilon \mapsto t_n$. In particular, the functor $\Lambda_p \odot -$ does not preserve the property of being p-torsion free.

There are two ways to approach this. One is to view $\mathbb{Z}[\varepsilon]/(\varepsilon^2)$ as a coequalizer of polynomial rings, for instance

$$\mathbb{Z}[x] \underset{x \mapsto 0}{\overset{x \mapsto \varepsilon^2}{\rightrightarrows}} \mathbb{Z}[\varepsilon] \longrightarrow \mathbb{Z}[\varepsilon]/(\varepsilon^2)$$

Then since $\Lambda_p \odot -$ preserves coequalizers (why?), we have a coequalizer diagram

$$\Lambda_p \odot \mathbb{Z}[x] \underset{x \mapsto 0}{\overset{x \mapsto \varepsilon^2}{\rightrightarrows}} \Lambda_p \odot \mathbb{Z}[\varepsilon] \longrightarrow \Lambda_p \odot \mathbb{Z}[\varepsilon]/(\varepsilon^2)$$

of Λ_p-rings. Since the map $\Lambda_p \to \Lambda_p \odot \mathbb{Z}[y]$, $f \mapsto f \odot y$, is an isomorphism, this diagram can be rewritten as

$$\Lambda_p \underset{e \mapsto 0}{\overset{e \mapsto e^2}{\rightrightarrows}} \Lambda_p \longrightarrow \Lambda_p \odot \mathbb{Z}[\varepsilon]/(\varepsilon^2).$$

So $\Lambda_p \odot \mathbb{Z}[\varepsilon]/(\varepsilon^2)$ is a quotient of $\Lambda_p = \mathbb{Z}[\theta_0, \theta_1, \dots]$ modulo a certain ideal, and the purpose of this exercise is to identify it.

Alternatively, a map $\Lambda_p \odot \mathbb{Z}[\varepsilon]/(\varepsilon^2) \to R$ is equivalent to a map $\mathbb{Z}[\varepsilon]/(\varepsilon^2) \to W(R)$, which is equivalent to a square-zero Witt vector. So you could just determine directly the conditions for a Witt vector a to satisfy $a^2 = 0$ in terms of its Witt components.

(3) Solve the previous question using the δ-generators instead of the θ-generators. More precisely, describe $\Lambda_p \odot \mathbb{Z}[\varepsilon](\varepsilon^2)$ as a quotient of $\mathbb{Z}[e, \delta, \dots] = \Lambda_p$ instead of $\mathbb{Z}[\theta_0, \theta_1, \dots]$.

(4) Consider the forgetful functor from the category of p-torsion-free Λ_p-rings to the category of p-torsion-free rings. As shown in lecture, this functor is comonadic and representable. Show it is also monadic. Can you extend this to a general context $(A, \mathfrak{p}, F)$?