Witt vectors, lambda-rings, and arithmetic jet spaces. Copenhagen, 2016.

Exercises for week 5.

(1) Show that the Verschiebung map $V_p\colon W(R) \to W(R)$ descends to a map $W_n(R) \to W_{n+1}(R)$ (also denoted V_p) and induces an exact sequence

$$0 \longrightarrow W_n(R) \xrightarrow{V_p} W_{n+1}(R) \longrightarrow R \longrightarrow 0.$$

Use this to show that the canonical map $W_n(R)[1/p] \to W_n(R[1/p])$ is an isomorphism.

(2) Show that the m-th ghost component map $w_m\colon W_n(R) \to R$ is integral, which is to say that every element of the codomain R satisfies a monic polynomial with coefficients in the image of w_n. Using this, show that the ghost map $W_n(R) \to R^{[0,n]}$ is integral. (Hint: recall that the sum of two integral elements is integral.)

(3) Prove the universal identity $V_p(x)V_p(y) = pV_p(xy)$. Use this to show that the kernel of the map $\mathbb{F}_p \otimes W_n(R) \to \mathbb{F}_p \otimes R$ is nilpotent. Can you find a similar formula for $V_p^i(x)V_p^j(y)$?

(4) Combine the questions above to show that the ghost map induces an integral morphism

$$\coprod_{[0,n]} \operatorname{Spec} R = \operatorname{Spec} R^{[0,n]} \longrightarrow \operatorname{Spec} W_n(R)$$

of schemes, which is an isomorphism away from p. Show that modulo p, this morphism can be identified with the projection

$$\coprod_{[0,n]} \operatorname{Spec} \mathbb{F}_p \otimes R \longrightarrow \operatorname{Spec} \mathbb{F}_p \otimes R,$$

at least if we work modulo nilpotent elements. Conclude that the ghost map

$$\coprod_{[0,n]} \operatorname{Spec} R \longrightarrow \operatorname{Spec} W_n(R)$$

is surjective. Thus one might say that, as a topological space, $\operatorname{Spec} W_n(R)$ looks like $n+1$ copies of $\operatorname{Spec} R$ glued together modulo p. Draw a picture of $\operatorname{Spec} W_1(\mathbb{Z}[x])$.

(5) For any $a \in R$, show that the canonical map $W_n(R)[1/[a]] \to W_n(R[1/a])$ is an isomorphism, where $[a] = (a, 0, 0, \ldots, 0)_\delta \in W_n(R)$ denotes the Teichmüller representative of a.