Witt vectors, lambda-rings, and arithmetic jet spaces. Copenhagen, 2016.

Exercises for week 4.

(1) Consider the composition ring $\Lambda'_p = \mathrm{colim}_{n,\psi} \Lambda_{p,n}$ as defined in the lecture. There we considered two elements $a, b \in \Lambda'_p$, where a comes from $e \in \Lambda_{p,1}$ and b comes from $-\delta \in \Lambda_{p,1}$. So formally we could write $a = \psi^{\circ -1}$, $b = -\psi^{\circ -1} \circ \delta$. Then a and b satisfy the relation $e + pb = a^p$, and a is $\mathbb{Z}$-algebra-like (in the sense of ex. 2, q. 4), and b has the Leibniz rules

$$
\begin{aligned}
b(x+y) &= b(x) + b(y) + \sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} a(x)^i a(y)^{p-i} \\
b(xy) &= a(x)^p b(y) + b(x) a(y)^p + p b(x) b(y) \\
b(0) &= 0 \\
b(1) &= 0.
\end{aligned}
$$

The converse is also true—any two such operators a and b on a ring come from a unique Λ'_p-structure. Give a completely detailed proof of this.

(2) Let S denote the subring of the product ring $\mathbb{Z}^{\mathbb{N}}$ consisting of all $\langle a_0, a_1, \ldots \rangle$ such that $a_{n+1} \equiv a_n \bmod p^{n+1}$ for all $n \geqslant 0$. In lecture it was stated that the ghost map $W(\mathbb{Z}) \to \mathbb{Z}^{\mathbb{N}}$ (which is injective) has image S, and hence that the ghost map induces an isomorphism $W(\mathbb{Z}) \to S$. Give a direct proof of this as follows:

First, show that S is the largest sub-ψ-ring of $\mathbb{Z}^{\mathbb{N}}$ on which ψ is a Frobenius lift; in other words, S is the largest sub-ψ-ring which is also a δ-ring. Then show that S satisfies the universal property of $W(\mathbb{Z})$, namely that for any δ-ring R, any ring map $R \to \mathbb{Z}$ lifts uniquely to a δ-morphism $R \to S$.

Your proof should use as little of the material developed in the class as possible.

(3) Let Λ'_p be as in question 1, and let W' denote the associated Witt vector functor. Determine $W'(\mathbb{F}_p)$ and $W'(\mathbb{Z})$. (They are both isomorphic to familiar concrete rings. Hint: Use the concrete descriptions of $W_n(\mathbb{F}_p)$ and $W_n(\mathbb{Z})$.) Are there any nonzero $\mathbb{F}_p$-algebras that admit a Λ'_p-ring structure? (Hint: Use the concrete description of $W'(\mathbb{F}_p)$.)

(4) The functor W preserves surjections, simply because $W(R) = R^{\mathbb{N}}$, as a set-valued functor. Does W' preserve surjections?

(5) In lecture it was shown that the map $W(\mathbb{Z}_p) \to \mathbb{Z}_p$ sending a Witt vector with ghost components $\langle a_0, a_1, \ldots \rangle$ to its p-adic limit $\lim_n a_n$ induces a ring map $W(\mathbb{F}_p) \to \mathbb{Z}_p$ and that this map is an isomorphism.

Prove the following more general version of this. Let F be an extension of $\mathbb{F}_p$ of degree d. The theory of local fields show there is a unique (up to unique isomorphism) complete discrete valuation ring R together with an isomorphism $R/pR \xrightarrow{\sim} F$, and that R has a unique endomorphism σ lifting the Frobenius map. (For example, if $d = 1$, then $F = \mathbb{F}_p$ and $R = \mathbb{Z}_p$. If $d = 2$ and $p \equiv 3 \bmod 4$, then $F = \mathbb{F}_p[i]$ and $R = \mathbb{Z}_p[i]$.) Find, as above, a morphism $W(R) \to R$ factoring through $W(R) \to W(R/pR) = W(F)$ to an isomorphism $W(F) \to R$.

Can you generalize this to where F is allowed to be an arbitrary perfect field of characteristic p?