Witt vectors, lambda-rings, and arithmetic jet spaces. Copenhagen, 2016.

Exercises for week 3.

(1) Let P be a composition k-algebra. Prove that the k-algebra maps

$$\Delta^+, \Delta^\times : P \to P \otimes_k P,$$

and $\varepsilon^+, \varepsilon^\times : P \to k$ are in fact P-ring maps. (What is the P-ring structure on k?)

(2) Let P be a composition k-algebra. An element $f \in P$ is said to be *additive* if $\Delta^+(f) = f \otimes 1 + 1 \otimes f$. Using question 1, show that f is additive if and only if on every P-ring, the operator is induces is additive in the usual sense. Let A_P denote the set of additive elements of P. Show that A_P has the structure of a k-algebra (possibly noncommutative), where addition is addition in P and where multiplication is composition in P.

(3) Let H be a cocommutative bialgebra over a field k of characteristic 0. Let $P = \mathrm{Sym}_k(H)$, with the usual composition structure. Show that the canonical inclusion $H \to P$ is a bijection onto A_P. Show that this is false if k has positive characteristic, provided $H \neq 0$.

(4) Consider the composition ring $\Lambda'_p = \mathrm{colim}_{n,\psi} \Lambda_{p,n}$ as defined in the lecture. There we considered two elements $a, b \in \Lambda'_p$, where a comes from $e \in \Lambda_{p,1}$ and b comes from $-\delta \in \Lambda_{p,1}$. So formally we could write $a = \psi^{\circ -1}$, $b = -\psi^{\circ -1} \circ \delta$. Then a and b satisfy the relation $e + pb = a^p$, and a is $\mathbb{Z}$-algebra-like (in the sense of ex. 2, q. 4), and b has the Leibniz rules

$$\begin{aligned} b(x+y) &= b(x) + b(y) + \sum_{i=1}^{p-1} \frac{1}{p} \binom{p}{i} a(x)^i a(y)^{p-i} \\ b(xy) &= a(x)^p b(y) + b(x) a(y)^p + p b(x) b(y) \\ b(0) &= 0 \\ b(1) &= 0. \end{aligned}$$

The converse is also true—any two such operators a and b on a ring come from a unique Λ'_p-structure. Give a completely detailed proof of this.

Are there any nonzero $\mathbb{F}_p$-algebras that admit a Λ'_p-ring structure?