Witt vectors, lambda-rings, and arithmetic jet spaces. Copenhagen, 2016.

Exercises for week 2.

(1) Find the formula for θ_2 in terms of $\delta^{\circ 0}, \delta^{\circ 1}, \delta^{\circ 2}$ and the formula for $\delta^{\circ 2}$ in terms of $\theta_0, \theta_1, \theta_2$. Use these formulas to find determine the relation between the Witt components and the δ-components on Witt vectors of length 2.

(2) *Comparison with other approaches.* It is most common to develop the theory of Witt vectors using the Witt components instead of the δ-components, as follows. As sets, we define $W(R) = R^{\mathbb{N}}$. Then consider the set map $w_R \colon W(R) \to R^{\mathbb{N}}$ defined by

$$w_R \colon (x_0, x_1, \dots) \mapsto \langle x_0, x_0^p + px_1, \dots, \sum_{i=0}^{n} p^i x_i^{p^{n-i}}, \dots \rangle.$$

It is functorial in R. (The polynomials on the right-hand side are called the *Witt polynomials.*) Then the standard treatments then prove the following facts from first principles.

(a) There is a unique ring structure on $W(R)$ which is functorial in R and has the property that the maps $w_R \colon W(R) \to R^{\mathbb{N}}$ become a ring homomorphisms, where the target $R^{\mathbb{N}}$ is understood to have the product ring structure.

(b) Let $F'_R : R^{\mathbb{N}} \to R^{\mathbb{N}}$ denote the shift map
$$F'_R \colon \langle y_0, y_1, \dots \rangle \mapsto \langle y_1, y_2, \dots \rangle.$$
Then there are unique maps $F_R : W(R) \to W(R)$, functorial in R, such that $w_R(F_R(x_0, x_1, \dots)) = F'_R(w_R(x_0, x_1, \dots))$ for all $(x_0, x_1, \dots) \in W(R)$. Further, each F_R is a ring homomorphism lifting the Frobenius map: $F_R(x) \equiv x^p \bmod pW(R)$.

(c) Suppose that R is p-torsion free and has a ring endomorphism $\psi \colon R \to R$ lifting the Frobenius map, and let S be a ring. Then any ring homomorphism $g \colon R \to S$ lifts uniquely to a ring homomorphism $\tilde{g} \colon R \to W(S)$ commuting with the Frobenius maps (i.e. $\tilde{g} \circ \psi = F_S \circ \tilde{g}$). The projection $W(S) \to S$ is understood to be the map $(x_0, \dots) \mapsto x_0$.

The first two facts are due to Witt; the third is due to Cartier, I believe, but he calls it the Dieudonné–Dwork lemma.

Show how these results follow from Joyal's theorem
$$\mathbb{Z}[\dots, \theta_n, \dots] \xrightarrow{\sim} \mathbb{Z}[\dots, \delta^{\circ n}, \dots].$$

Conversely, show that Joyal's theorem (and hence that $W(R)$ as defined above is the co-free δ-ring on R) follows almost completely formally from the results above. You may assume the formal, category-theoretic properties of δ-rings, as given in lecture.

Thus one could say that Joyal's theorem is formally equivalent to the three facts above. In particular, it contains all the congruence-theoretic information necessary to prove Witt's two results, which is why one might also call it Witt's theorem.

The weakness of the standard approach is that the three facts above are non-formal, in that they involve slightly subtle arguments with p-adic congruences, but they are required to get even the most basic part theory off the ground—for instance even to define the ring structure on $W(R)$. Not only is their content not formal, but the facts themselves do not seem so natural, at least to a newcomer. The point of the development in the lecture (which, as mentioned, is due to Joyal) is that with the δ-coordinates, we can go quite far in a formal and natural way. All arguments with congruences are quarantined in Joyal's theorem, which is in fact only needed to make the Witt coordinates. And their role is now reduced to an explicit computational device which is convenient for certain purposes.

It is worth comparing the development in lecture with most accounts in the literature, which are filled with formulas. Indeed, if one defines Witt vectors using the Witt polynomials, then any argument about them will ultimately have to be given in those terms.

(3) Let M be a monoid, and let $k[M]$ denote the polynomial k-algebra on the set underlying M. As discussed in lecture, $k[M]$ is a k-plethory. Describe the structure maps Δ^+, $\Delta^\times$, ε^+, $\varepsilon^\times$, and $\circ$ explicitly (including showing that they're well defined), and give a complete proof that $k[M]$ is a k-plethory.

(4) Let P be a k-plethory. Let us say that an element f *acts as a k-algebra map* (or is *k-algebra-like*) if for any P-ring A, the map $A \to A$ given by $a \mapsto f(a)$ is a k-algebra homomorphism. For example, $\psi \in \Lambda_p$ is such an element but $\delta \in \Lambda_p$ is not. Show this is equivalent to requiring $\Delta^+(f) = f \otimes 1 + 1 \otimes f$, $\Delta^\times(f) = (f \otimes 1) \times (1 \otimes f) = f \otimes f$, and $f \circ c = c$ for all $c \in k$. (This formulation has the advantage of not quantifying over all P-rings.) Prove that the set of k-algebra-like elements is a monoid under composition.

Define what it means for an element f to act as a k-derivation, and prove that the set of such elements is a Lie algebra over k.

(5) Base change of plethories. Let P be a k-plethory, let k' be a P-ring, and write $P' = k' \otimes_k P$. Show that P' has a natural k'-plethory structure. More precisely, there is a k'-plethory structure on P' such that any action of P on a k'-algebra compatible with the action on k' extends uniquely to an action of P'. What if we only require k' to be a k-algebra?

(6) Show that there is a plethory B_p such that B_p-rings are δ-rings on which the Frobenius map $\psi = e^p + p\delta$ is the identity. Show there is a natural surjective map $\Lambda_p \to B_p$. Determine explicit generators for its kernel. Determine $\mathbb{Z}[1/p] \otimes_{\mathbb{Z}} B_p$ explicitly. (It is a simple ring.) Now let R be a p-torsion free ring. Determine $W_P(R)$ explicitly. (Some people call a B_p-ring the *p-typical binomial plethory*.)

(7) Do the same thing for δ-rings on which the Frobenius map is idempotent instead of the identity.

(8) For some k-plethories P, a P-ring structure is actually a *property* (rather than only a structure). In other words, the forgetful functor from P-rings to k-algebras is full and faithful (rather than only faithful). Such a plethory is said to be *idempotent*, following Jesse Elliott. Show that a plethory is idempotent if and only if the composition map $P \odot P \to P$ is an isomorphism. Show that the category of perfect $\mathbb{F}_p$-algebras (i.e. those on which the Frobenius map $x \mapsto x^p$ is a bijection) is the category of P-rings for an $\mathbb{F}_p$-plethory P (necessarily idempotent). Describe $P \odot R$ and $W_P(R)$

in concrete terms not involving plethories or birings. Which of the other plethories on this exercise sheet are idempotent?

(9) Consider the functor W' where $W'(R) = \lim_\psi W(R)$ is the inverse limit of the system
$$\cdots \xrightarrow{\psi} W(R) \xrightarrow{\psi} W(R) \xrightarrow{\psi} W(R).$$
Show that W' is the Witt vector functor of a $\mathbb{Z}$-plethory, which we might denote $\Lambda_p\langle\psi^{\circ-1}\rangle$. Say what an action of $\Lambda_p\langle\psi^{\circ-1}\rangle$ on a ring is in concrete terms. Explain why $\Lambda_p\langle\psi^{\circ-1}\rangle$ is a good notation.

(10) Do the same as in the previous question for the functor sending R to the inverse limit
$$\cdots \xrightarrow{\psi} W_2(R) \xrightarrow{\psi} W_1(R) \xrightarrow{\psi} W_0(R).$$

(11) *Exponentials in positive characteristic.* (Magnus Carlson) Consider the following structure on an $\mathbb{F}_p$-algebra R: two functions $n, x : R \to R$, where n is a ring endomorphism which takes values in the Frobenius-invariant subring of R, and where x satisfies the laws $x(a + b) = x(a)x(b)$, $x(0) = 1$, $x(1) = 1$, and $x(ab) = x(a)^{n(b)}x(b)^{n(a)}$.

(The last notation requires some explanation. First, observe that $x(a)$ is always a p-th root of unity. The reason, then, that $x(a)^{n(b)}$ and similar expressions make sense is that if we view $x(a)$ and $n(b)$ as global sections of $\mathcal{O}_{\operatorname{Spec} R}$, then $n(b)$ is a locally constant element of $\mathbb{F}_p$, and $x(a)$ is a p-th root of unity; so $x(a)^{n(b)}$ has a well-defined meaning over any sufficiently fine cover, and hence globally. For example, if R is an integral domain, then its spectrum is connected, and so $n(a)$ is simply an element of $\mathbb{F}_p$. More generally, suppose the set S of connected components of $\operatorname{Spec} R$ is finite. Then n can be viewed as a ring map $R \to \mathbb{F}_p^S$. In other words, giving n is equivalent to endowing each connected component with an $\mathbb{F}_p$-valued point. Then $n(a)$ is simply the value of the function a at the point given by n.)

A morphism of two rings with such structure is of course a ring map commuting with n and x. Show that such a structure is equivalent to the action of a plethory P, and determine P explicitly.

Note that since x satisfies $x(a+b) = x(a)x(b)$, it is some kind of exponential map in characteristic p. It follows from a recent result of Magnus Carlson's that exponential maps don't exist in plethories in characteristic 0, perhaps (one might say) because there is no axiom for the exponential of a product. Amazingly however they do exist in positive characteristic.