Witt vectors, lambda-rings, and arithmetic jet spaces. Copenhagen, 2016. Exercises for week 1.

Exercises 8–10 are somewhat more elaborate and will not be discussed in the exercise class.

(1) Determine the Leibniz rules for $\delta^{\circ 2}(x+y)$ and $\delta^{\circ 2}(xy)$ when $p=2$.

(2) Prove that the only $\mathbb{F}_p$-algebra admitting a p-derivation is the zero ring. More generally, prove that the only $\mathbb{Z}/p^n\mathbb{Z}$-algebra admitting a p-derivation is the zero ring.

(3) Give an example of a δ-ring which is not p-torsion free. Prove that every p^n-torsion element in a δ-ring is nilpotent.

(4) Show that $\mathbb{Z}[x]/(x^n-1)$ admits a unique p-derivation. Now let $\mathbb{Z}[\zeta_n]$ denote the subring of $\mathbb{C}$ generated by a primitive n-th root of unity ζ_n. For which primes p does $\mathbb{Z}[\zeta_n]$ admit a p-derivation? Are these p-derivations unique?

(5) (a) Let C denote the category of rings with Frobenius lift. [So the objects of C are rings R (commutative) together with a ring map $\psi\colon R \to R$ such that the induced map $R/pR \to R/pR$ is the Frobenius map $x \mapsto x^p$. The morphisms are ring homomorphisms which commute with the ψ maps.] Show that C has all colimits and all products and that the forgetful functor to the category of rings preserves them. But show that C does not have equalizers.

(b) This is an open-ended follow-up to part (a), and which I don't know the answer to. The introduction of the concept of δ-rings given in the lecture can be viewed as having two gaps in motivation. One is that p-torsion-free rings have a special role, and it's not clear why they should. The second is that, even given the special role of p-torsion-free rings, the axioms for δ, while they seem natural enough, are not produced in a systematic way. So it would be nice to have a general category-theoretic machine which when given the category of rings with Frobenius lift as an input, would output the category of δ-rings. Is it possible to write down such a machine which resolves these gaps in motivation in a satisfying way?

(6) Consider Witt vectors $x=(x_0,x_1,\dots)$ and $y=(y_0,y_1,\dots)$, where all x_i and y_i lie in a square-zero ideal. Find simple formulas for $x+y$ and xy. You can use the Witt coordinates or the δ-coordinates, whichever you prefer.

(7) Consider a Witt vector whose first n components are $(0,\dots,0,x)$. Determine its first n ghost components. You can use the Witt coordinates or the δ-coordinates, whichever you prefer.

(8) The purpose of this exercise to give some descriptions of $W(R)$ which have a more linear feel.

(a) Let R be a p-torsion-free δ-ring. Prove that the ghost map $W(R) \to R^\infty$ induces a bijection

$$W(R) = \{\langle a_0, a_1, \dots\rangle \in R^\infty \mid \psi(a_n) \equiv a_{n+1} \bmod p^{n+1}R\}$$

(b) We can give another interpretation of part (a). Given the ghost components $\langle a_0, a_1, \dots\rangle$ of a Witt vector, let $b_n = (\psi(a_n)-a_{n+1})/p^{n+1}$. Then the map $W(R) \to R^\infty$ sending a Witt vector to the vector $[b_0, b_1, \dots]$ is a bijection. One might call the components b_i the *necklace components* of the original Witt vector. (I use square brackets only to distinguish the necklace components from other components, such as the ghost components.) The necklace components have the disadvantage that they are only defined for δ-rings, but they have the advantage that the ring operations on $W(R)$ become very simple when expressed in terms of them. Do this. Is this also valid for when R is a δ-ring which is not p-torsion free?

(c) Use the previous question to give a concrete description of the ring of Witt vectors of a polynomial ring $\mathbb{Z}[X]$, where X is a set. Your answer should involve only basic concepts from the theory of polynomials. No p-derivations or anything like that.

(d) Let R be an arbitrary ring, let $\mathbb{Z}[R]$ denote the free polynomial ring $\mathbb{Z}[x_r \mid r \in R]$ (not the monoid algebra!), and let $\mathbb{Z}[R] \to R$ denote the map given by $x_r \mapsto r$. This realizes $W(R)$ as a quotient of $W(\mathbb{Z}[R])$ in a canonical way. Use this to give an alternative definition of $W(R)$ for general rings R.

(9) Let k be a perfect field of characteristic p. Let R be a complete discrete valuation ring with maximal ideal p and residue field k. (Such an R always exists. Why?) The following argument shows that $W(k)$ is canonically isomorphic to R.

Show that R has a unique Frobenius lift ψ and that ψ is an isomorphism. Let a be a Witt vector in $W(R)$, and let $\langle a_0, a_1, \dots \rangle$ denote its ghost components. Show that the sequence $\psi^{-n}(a_n) \in R$, where ψ^{-n} denotes the inverse of the n-th iterate of ψ, is a Cauchy sequence in the p-adic topology on R and hence has a limit. Show that the map $W(R) \to R$ sending a Witt vector a to this limit is a ring homomorphism and factors (uniquely) through the quotient map $W(R) \to W(k)$ to induce an isomorphism $W(k) \to R$.

Conclude that $W(\mathbb{F}_p) = \mathbb{Z}_p$. More generally, if k is a finite field, then $W(k)$ is the ring of integers in the maximal unramified extension of $\mathbb{Q}_p$ with residue field k.

(10) Show that $W(\mathbb{F}_p[x])$ is canonically isomorphic to the set of formal series $\sum_{n \in \mathbb{N}[1/p]} a_n X^n$, where $a_n \in \mathbb{Z}_p$ and $\mathrm{den}(n) \mid a_n$ and the a_n approach 0 in the sense that for any $m > 0$ all but finitely many a_n are divisible by p^m. Here $\mathbb{N}[1/p]$ denotes the set of rational numbers of the form a/p^b for integers $a, b \geqslant 0$, and $\mathrm{den}(n)$ denotes the denominator of n, which is to say p^b where b is minimal. Can you generalize this to $W(\mathbb{F}_p[x, y])$?