Lectures on Partial Differential Equations

January 14, 2005

1 Course Logistics

The subject of partial differential equations (PDE) encompasses several sub-branches of mathematics. Two of the main sub-categories correspond to the study of Elliptic and Hyperbolic differential equations. The first three weeks of this course will be dedicated to the study of Elliptic equations, while the last week will be used to go over some aspects of hyperbolic equations.

- Text: *Partial Differential Equations of Second Order* by David Gilbard and Neil Trudinger.
- Outline:

Week 1: Potential Theory and Maximum Principles.

Week 2: Sobolev Spaces.

Week 3: Applications to General Equations.

2 Preliminaries

2.1 Notation

2.1.1 Set Notation

Ω will signify an open subset of Euclidean n-space, $\mathbb{R}^n$. A *domain* is understood to be a a connected, open set. In addition, we use the following standards of notation:

$$|\Omega| = \text{volume of } \Omega = \text{Lebesgue measure of } \Omega.$$
$$S \subset \mathbb{R}^n,\ \partial S = \text{ boundary of } S,\ \overline{S} = \text{ closure of } S.$$

Balls are extremely important sets in PDE theory and thus warrant their own notation. $B_R(y)$ signifies an open ball of radius R and center y. Correspondingly ω_n refers to the volume of the unit ball in $\mathbb{R}^n$. The following relation's derivation is left as an exercise,

$$\omega_n = \frac{2\pi^{n/2}}{n \cdot \Gamma\left(\frac{n}{2}\right)}.$$

Aside: The following is a solution to the above exercise. First, recall that

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}.$$

With that, we have

$$\begin{aligned}
\pi^{n/2} &= \prod_{i=1}^{n} \int_{-\infty}^{\infty} e^{-x_i^2}\, dx_i \\
&= \int_{\mathbb{R}^n} e^{-\sum_{i=1}^{\infty} x_i^2}\, dV \\
&= \oint \int_0^{\infty} r^{n-1} e^{-r^2}\, dr d\Omega \\
&= \frac{1}{2} \int_0^{\infty} z^{n/2-1} e^{-z}\, dz \cdot \oint d\Omega \\
&= \frac{\Gamma\left(\frac{n}{2}\right)}{2} \cdot \oint d\Omega.
\end{aligned}$$

Thus, we have ascertained that

$$\oint d\Omega = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}.$$

To derive the integral definition of the Gamma function, we note that

$$\int_0^{\infty} e^{-tx}\, dt = \frac{1}{x}.$$

Taking n derivates with respect to x of both sides then yields

$$\frac{d^n}{dx^n} \int_0^{\infty} e^{-tx}\, dt = \int_0^{\infty} t^n e^{-tx}\, dt = \frac{n!}{x^{n-1}}.$$

Setting $x = 1$ gives

$$\int_0^{\infty} t^n e^{-t}\, dt = n! = \Gamma(n+1).$$

With that in mind, we calculate

$$\begin{aligned} V_n(R) &= \oint \int_0^R r^{n-1}\, dr d\Omega \\ &= \frac{R^n}{n} \cdot \oint d\Omega \\ &= \frac{2\pi^{n/2}}{n \cdot \Gamma\left(\frac{n}{2}\right)} R^n. \end{aligned}$$

This completes the exercise.

2.1.2 Notation for Derivatives

Taking $x \in \mathbb{R}^n$, i.e. $x = (x_1, \ldots, x_n)$, we have the following standard notation for the partial derivative:

$$D_i = \frac{\partial u}{\partial x_i},$$

and for the gradient

$$Du = (D_1 u, \ldots, D_n u)(= \nabla u).$$

Note that ∇u is classical notation as Du will often indicated a weak gradient. Correspondingly, we have for second order derivatives

$$D_{ij} u = \frac{\partial^2 u}{\partial x_i \partial x_j}.$$

Next, we define the *Hessian Matrix* of u as

$$D^2 u = [D_{ij} u],$$

where the indices i and j correspond to the rows and columns of the Hessian matrix respectively. A convenient alternative notation that sometimes lends itself to confusion is the following:

$$u_i = D_i u, \;\; u_{ij} = D_{ij} u.$$

The final piece of notation that we will go over is that of *multi-index notation*. Consider a vector $\beta = (\beta_1, \ldots, \beta_n)$ with $0 \leq \beta_i \in \mathbb{Z}$. In the familiar way, we define the magnitude of the vector as

$$|\beta| = \sum_{i=1}^{n} \beta_i^2.$$

With that, we write

$$D^\beta u = \frac{\partial^{|\beta|} u}{\partial x_i^{\beta_1} \cdots \partial x_n^{\beta_n}}.$$

Example 1. *If $n = 3$ and $\beta = (1, 2, 0)$, we have*

$$D^\beta u = \frac{\partial^3 u}{\partial x_1 \partial x_2^2}.$$

Remark: Multi-index notation really becomes a necessity when dealing with equations with order > 2.

2.2 Classical Function Spaces

Now, we will go through the definitions of classical functions spaces used in PDE theory. First, we have spaces of continuous functions:

$$\begin{aligned} C^0(\Omega) &= \{\text{functions continuous in } \Omega\} \\ C^0(\overline{\Omega}) &= \left\{\text{functions continuous in } \overline{\Omega}\right\}. \end{aligned}$$

Next, we have spaces of functions with continuous classical derivatives:

$$\begin{aligned} k \geq 0,\ C^k(\Omega) &= \{\text{functions whose deriv. up to and including} \\ &\quad\ \text{order } k \text{ are continuous in } \Omega\} \\ C^k(\overline{\Omega}) &= \{\text{functions whose deriv. of order} \leq k \text{ have} \\ &\quad\ \text{continuous extensions to } \overline{\Omega}\}. \end{aligned}$$

Remark: Wording of the last definition is a bit different than saying the derivatives of order $\leq k$ continuous on $\overline{\Omega}$; this is actually a bit stronger considering certain pathological domains.

The *Support* of u is defined as

$$\operatorname{supp} u = \overline{\{x \in \mathbb{R}^n \mid u(x) \neq 0\}}.$$

Correspondingly, we define

$$C_0^k(\Omega) = \{\text{functions in } C^k(\Omega) \text{ having compact support in } \Omega\}.$$

In the case where $k = \infty$, we have get a very important function space from the above definition, namely

$$C_0^\infty(\Omega) = \{\text{infinitely differentiable functions with compact support in } \Omega\}.$$

Specifically, $C_0^\infty(\Omega)$ is the space of all test functions, whose importance will be demonstrated later.

3 Weak Maximum Principle

To begin this section, we introduce the *Laplacian* differential operator:

$$\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}, \; u \in C^2(\Omega).$$

Note that Δu is sometimes written as $\operatorname{div} \cdot \nabla u$ in classical notation. Coupled to this we have the following definition

Definition 3.1. *$u \in C^2(\Omega)$ is called* harmonic *in Ω if and only if $\Delta u = 0$ in Ω.*

Example 1. *$u(x_1, x_2) = x_1^2 - x_2^2$ is a nonlinear harmonic function in $\mathbb{R}^2$.*

In addition to the above, we have the next definition.

Definition 3.2. *$u \in C^2(\Omega)$ is called* subharmonic*(*superharmonic*) if and only if $\Delta u \geq 0 (\leq 0)$ in Ω.*

Remark: It is obvious, due to the linearity of the Laplacian, that if u is subharmonic, then $-u$ is superharmonic and vice-versa.

Example 2. *If $x \in \mathbb{R}^n$, then $\Delta |x|^2 = 2n$. Thus, $|x|^2$ is subharmonic in $\mathbb{R}^n$.*

Now, we come to our first theorem

Theorem 3.3 (Weak Maximum Principle for the Laplacian). *Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ with Ω bounded in $\mathbb{R}^n$. If $\Delta u \geq 0$ in Ω, then*

$$u \leq \max_{\partial\Omega} u \; \left(\iff \max_{\overline{\Omega}} u = \max_{\partial\Omega} u \right). \tag{1}$$

Conversely, if $\Delta u \leq 0$ in Ω, then

$$u \geq \min_{\partial\Omega} u \; \left(\iff \min_{\overline{\Omega}} u = \min_{\partial\Omega} u \right). \tag{2}$$

Consequently, if $\Delta u = 0$ in Ω, then

$$\min_{\partial\Omega} u \leq u \leq \max_{\partial\Omega} u. \tag{3}$$

Proof: We will just prove the subharmonic result, as the superharmonic case is proved the same way. Given u subharmonic, take $\epsilon > 0$ and define $v = u + \epsilon|x|^2$, so that $\Delta v = \Delta u + 2\epsilon n > 0$ in Ω. If v takes attains a maximum in Ω, we have $\Delta v \leq 0$ at that point, a contradiction. Thus, by the compactness of Ω, we have

$$v \leq \max_{\partial\Omega} v.$$

Finally, we take $\epsilon \to 0$ in the above to get the result. ■

Corollary 3.4 (Uniqueness). *Take* $u, v \in C^2(\Omega) \cap C^0(\overline{\Omega})$. *If* $\Delta u = \Delta v = 0$ *in* Ω *and* $u = v$ *on* $\partial\Omega$, *then* $u = v$ *in* Ω.

Before moving onto more general elliptic equations, we briefly remind ourselves of the *Dirichlet Problem*, which takes the following form.

$$\begin{aligned} &\Delta u = 0 \text{ in } \Omega \\ &u = \phi \text{ on } \partial\Omega \text{ for some } \phi \in C^0(\partial\Omega) \end{aligned}$$

4 Linear Elliptic Operators

Again we take $\Omega \subset \mathbb{R}^n$. We now consider the following differential operator:

$$Lu := \sum_{i,j=1}^{n} a^{ij} D_{ij} u + \sum_{i=1}^{n} b^i D_i u + cu,$$

where $a^{ij}, b^i, c : \Omega \to \mathbb{R}$. Now, we state the following.

Definition 4.1. *The operator* L *is said to be* elliptic(degenerate elliptic) *if* $\mathcal{A} := [a^{ij}] > 0 (\geq 0)$ *in* Ω.

Remarks:

i.) In the above definition $\mathcal{A} > 0$ means the minimum eigenvalue of the matrix $\mathcal{A}$ is > 0. More explicitly, this means that

$$\sum_{i,j=1}^{n} a^{ij} \xi_i \xi_j > 0, \; \forall \xi \in \mathbb{R}^n.$$

ii.) The last definition indicates that parabolic PDE (such as the Heat Equation) are really degenerate elliptic equations.

For convenience, we will be using the Einstein summation convention for repeated indicies. This means that in any expression where letters of indicies appear more than once, a summation is applied to that index from 1 to n. With that in mind, we can write our general operator as

$$Lu := a^{ij}D_{ij}u + b^i D_i u + cu.$$

Now, we move onto the weak maximum principle for our special case operator:

$$Lu = a^{ij}D_{ij}u.$$

Theorem 4.2 (Weak Maximum Principle). *Consider $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ and $f : \Omega \to \mathbb{R}$. If $Lu \geq f$, then*

$$u \leq \max_{\partial\Omega} u + \frac{1}{2}\left(\sup_{\overline{\Omega}} \frac{|f|}{\operatorname{Tr}\mathcal{A}}\right) \cdot (\operatorname{diam}\Omega)^2\,, \tag{4}$$

where $\operatorname{Tr}\mathcal{A}$ is the trace of the coefficient matrix $\mathcal{A}$ (i.e. $\operatorname{Tr}\mathcal{A} = \sum_{i=1}^n a^{ii}$).

Note: If $f \equiv 0$, the above reduces to the result of the first weak maximum principle we proved. Namely, $u \leq \max_{\partial\Omega} u$.

Proof: Without loss of generality we may translate Ω so that it contains the origin. Again, we consider an auxiliary function:

$$v = u + k|x|^2,$$

where k is an arbitrary constant to be determined later. Since the Hessian Matrix D^2u is symmetric, we may choose a coordinate system such that D^2u is diagonal. In this coordinate system, we calculate

$$Lv = Lu + 2ka^{ij}S_{ij} = Lu + 2k \cdot \operatorname{Tr}\mathcal{A},$$

where $S_{ii} = 1$ and $S_{ij} = 0$ when $i \neq j$. Now, suppose that v attains an interior maximum at $y \in \Omega$. From calculus, we then know that $D^2v(y) \leq 0$, which implies that $a^{ij}D_{ij} \leq 0$ as $\mathcal{A}$ is positive definite. Now, if we choose

$$k > \frac{1}{2}\sup_{\overline{\Omega}} \frac{|f|}{\operatorname{Tr}\mathcal{A}},$$

our above calculation indicates that $Lv > 0$; a contradiction. Thus,

$$v \leq \max_{\partial\Omega} v$$

which implies the result as Ω contains the origin (which indicates that $|x|^2 \leq (\operatorname{diam}\Omega)^2$). ■

4.1 Application

Now we will apply the weak maximum principle to the second most famous elliptic PDE, the *Minimal Surface Equation*:

$$(1+|Du|^2)\Delta u - D_i u D_j u D_{ij} u = 0.$$

This is a quasilinear PDE as only derivatives of the 1st order are multiplied against the 2nd order ones (a fully nonlinear PDE would have products of second order derivatives). We will now show that this is indeed an elliptic PDE. Recalling our special case $Lu = a^{ij} D_{ij} u$, we can rewrite the minimal surface equation in this form with

$$a^{ij} = (1+|Du|^2)S_{ij} + D_i u D_j u,$$

with S being as it was in the previous proof. Now, we calculate

$$\begin{aligned} a^{ij}\xi_i\xi_j &= (1+|Du|^2)S_{ij}\xi_i\xi_j - D_i u D_j u \xi_i \xi_j \\ &= (1+|Du|^2)|\xi|^2 - D_i u D_j u \xi_i \xi_j \\ &= (1+|Du|^2)|\xi|^2 - (D_i u \xi_i)^2 \\ &\geq (1+|Du|^2)|\xi|^2 - |Du|^2|\xi|^2 \\ &= |\xi|^2. \end{aligned}$$

Note: To get the third equality in the above, we simply relabeled repeated indicies. Since one sums over a repeated index, it's representation is a dummy index, whose relabeling does not affect the calculation.

From this calculation, we conclude this equation is indeed elliptic. Now, upon taking $f = 0$ and replacing u by $-u$, we can now apply the weak maximum principle to this equation.

4.2 Weak Maximum Principle for General Elliptic Equations

Now, we will go over the weak maximum principle for operators having all order of derivatives ≤ 2.

Theorem 4.3 (Weak Maximum Principle). *Consider $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$; $a^{ij}, b^i, c : \Omega \to \mathbb{R}$; $c \le 0$; and Ω bounded. If u satisfies the elliptic equation (i.e. $[a^{ij}] = \mathcal{A} > 0$)*

$$Lu = a^{ij} D_{ij} + b^i D_i u + cu \ge f, \tag{5}$$

then

$$u \le \max_{\partial\Omega} u^+ + C\left(n, \operatorname{diam}\Omega, \sup_{\overline{\Omega}} \frac{|b|}{\lambda}\right) \cdot \sup_{\overline{\Omega}} \frac{|f|}{\lambda}, \tag{6}$$

where

$$\lambda = \min_{|\xi|=1} a^{ij}\xi_i\xi_j \quad \textit{and} \quad u^+ = \max\{u, 0\}.$$

Note: If a^{ij} is not constant, then λ will almost always be non-constant on Ω as well.

<u>Consequences:</u>

- If $f = 0$, then $u \le \max_{\partial\Omega} u^+$.
- If $f = c = 0$, then $u \le \max_{\partial\Omega} u^+$.
- If $Lu \le f$ one gains a lower bound

  $$u \ge \max_{\partial\Omega} u^- - C \sup_{\overline{\Omega}} \frac{|f|}{\lambda},$$

  where $u^- = \min\{u, 0\}$.
- If we have the PDE $Lu = f$, then one has both an upper and lower bound:

  $$|u| \le \max_{\partial\Omega} |u| + C \sup_{\overline{\Omega}} \frac{|f|}{\lambda}.$$

- Uniqueness of the Dirichlet Problem. Recall, that the problem is as follows

  $$\begin{cases} \text{PDE:} \quad Lu = f & \text{in} \quad \Omega \\ \qquad\qquad u = \phi & \text{on} \quad \partial\Omega \end{cases}.$$

  So if one has $u, v \in C^2(\Omega) \cap C^0(\overline{\Omega})$ with $Lu = Lv$ in Ω, and $u = v$ on $\partial\Omega$, then $u \equiv v$ in Ω. The proof is the simple application of the weak maximum principle to $w = u - v$, which is obviously a solution of PDE as it's linear.

Proof of Weak Max. Principle: As in previous proofs, we analyze a particular auxiliary function to prove the result.

<u>Aside:</u> $v = |x|^2$ will not work in this situation as $Lv = 2 \cdot \mathrm{Tr}\, \mathcal{A} + 2b^i x_i + c|x|^2$; the $b^i x_i$ may be a large negative number for large values of $|x|$.

So, let us try

$$v = e^{\alpha(x,\xi)},$$

where ξ is some arbitrary vector. Also, we will consider $\Omega^+ = \{x \in \Omega \,|\, u > 0\}$ instead of Ω, for the rest of the proof. Clearly, this does not affect the result as we are seeking to bound u by $\max_{\partial\Omega} u^+$. With that, we have on Ω^+

$$\begin{aligned} L_0 u := a^{ij} D_{ij} u + b^i D_i u &\geq -cu + f \\ &\geq f. \end{aligned}$$

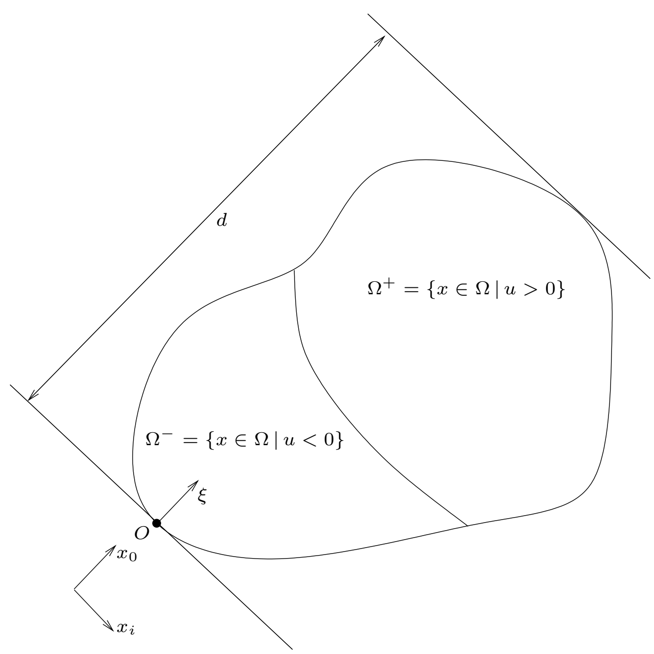

Figure 4.1:

Now, we wish to specify which vector ξ is. For the following, please refer to the figure. We first choose our coordinates so that the origin corresponds to a boundary point of Ω such that there exists a hyperplane through that point with Ω lying on one side of it. ξ is simply taken to be the unit vector normal to this hyperplane. For simplicity we rotate our coordinate system such that $\xi_i = 0$ for $i > 0$. Next, we calculate

$$\begin{aligned} L_0 v &= (\alpha^2 a^{ij}\xi_i\xi_j + \alpha b^i\xi_i)e^{\alpha(x,\xi)} \\ &= (\alpha^2 a^{ij}\xi_i\xi_j + \alpha b^i\xi_i)e^{\alpha(x,\xi)} \\ &\geq \alpha\lambda e^{\alpha(x,\xi)}, \end{aligned}$$

provided we choose $\alpha \geq \left(sup_{\overline{\Omega}}\frac{|b|}{\lambda} + 1\right)$.

Aside: The calculation for the determination of α is as follows:

$$\alpha^2 a^{ij}\xi_i\xi_j + \alpha b^i\xi_i \quad \geq \quad \alpha^2\lambda + \alpha b^i\xi_i.$$

So, $\alpha^2\lambda + \alpha b^i\xi_i \geq \alpha\lambda$ implies that

$$\begin{aligned} \alpha\lambda &\geq \lambda - b^i\xi_i \\ \Longrightarrow \alpha &\geq 1 - \frac{1}{\lambda}\sup_{\overline{\Omega}}|b| \\ \Longrightarrow \alpha &\geq 1 - \sup_{\overline{\Omega}}\frac{|b|}{\lambda}. \end{aligned}$$

For convenience, define $\tilde{b} = \sup_{\overline{\Omega}}\frac{|b|}{\lambda}$. So, clearly we have the desired inequality if we choose $\alpha \geq \tilde{b} + 1$.

Given our choice of ξ, it is clear that $(x, \xi) \geq 0$ for all $x \in \Omega$. Thus, one sees that $L_0 v \geq \alpha\lambda$. So, one can ascertain that

$$L_0(u^+ + kv) \geq f + \alpha\lambda k > 0 \text{ in } \Omega^+,$$

provided k is chosen so that

$$k > \frac{1}{\alpha}\sup_{\overline{\Omega}}\frac{|f|}{\lambda}.$$

Now, the proof proceeds as previous ones. Suppose $w := u^+ + kv$ has a positive maximum in Ω^+ at a point y. In this situation we have

$$[D_i j w(y)] \leq 0 \Longrightarrow a^{ij}D_{ij}w(y) \leq 0,\ b^i D_i w(y) = 0,$$

which implies $Lw \leq 0$; a contradiction. So, this implies the second inequality of the following:

$$\begin{aligned} u + \frac{e^0}{\alpha} \sup_{\overline{\Omega}} \frac{|f|}{\lambda} &\leq w \\ &\leq \max_{\partial\Omega^+} w \\ &\leq \max_{\partial\Omega^+} u^+ + \frac{e^{\alpha d}}{\alpha} \cdot \sup_{\overline{\Omega}} \frac{|f|}{\lambda} \\ &\leq \max_{\partial\Omega} u^+ + \frac{e^{\alpha d}}{\alpha} \cdot \sup_{\overline{\Omega}} \frac{|f|}{\lambda}, \end{aligned}$$

again where $\alpha = \tilde{b} + 1$ and d is the breadth of Ω in the direction of ξ. From the above, we finally get

$$u \leq \max_{\partial\Omega} u^+ + \frac{e^{\alpha d} - 1}{\alpha} \cdot \sup_{\overline{\Omega}} \frac{|f|}{\lambda}. \qquad \blacksquare$$

Remark: Given the above, one needs $\frac{|b|}{\lambda}$ and $\frac{|f|}{\lambda}$ bounded for the weak maximum principle to give a non-trivial result. Then one may apply the weak maximum principle to get uniqueness.

5 Poisson Integral Formula

The best possible scenario one can hope for when dealing in theoretical PDE is to actually have an analytic solution to the equation. Obviously, it is easier to analyze a solution directly rather than having to do so through its properties. Fortunately, this turns out to be possible for the Dirichlet problem for Laplace's equation in a ball.

To start off, let us consider some other examples of harmonic functions:

- Since Laplace's equation is a constant coefficient equation, we know that if $u \in C^3(\Omega)$ is harmonic, then so is all its partial derivatives. Indeed,

$$D_i(\Delta u) = 0 \iff \Delta(D_i u) = 0.$$

- There are also radially symmetric solutions to Laplace's equation. To derive these, let $r = |x|$ and $u(x) = g(r)$. We calculate

$$\begin{aligned} D_i u &= g' \cdot \frac{x_i}{|x|} = \frac{g'}{r} \cdot x_i \\ \Delta u &= D_{ii} u = n \cdot \frac{g'}{r} + \frac{g''}{r^2} \cdot x_i^2 - \frac{g'}{r^3} x_i^2 \\ &= n \cdot \frac{g'}{r} + g'' - \frac{g'}{r} \\ &= (n-1) \cdot \frac{g'}{r} + g'' \\ &= \frac{(r^{n-1} g')'}{r^{n-1}} = 0. \end{aligned}$$

This implies that $r^{n-1} g'(r) =$ constant; integration then yields

$$g = \begin{cases} C_1 r^{2-n} + C_2 & n > 2 \\ C_1 \ln r + C_2 & n = 2 \end{cases}.$$

These are all the radial solution of the homogeneous Laplace's equation outside of the origin. For simplicity we will take $C_1 = 1$ and $C_2 = 0$ in the above.

- It is easily verified that translating the above solutions so that y is taken to be the new origin, are also harmonic functions:

$$\begin{cases} |x-y|^{2-n} & n > 2 \\ C_1 \ln |x-y| & n = 2 \end{cases}.$$

Since these functions are symmetric in x and y, it's obvious the above are harmonic with respect to either x or y provided $x \neq y$. Taking a derivative of the above indicates that

$$\frac{x_i - y_i}{|x_i - y_i|^n}$$

is also harmonic. Next consider

$$\begin{aligned} V(x,y) &= \frac{|y|^2 - |x|^2}{|x-y|^n} = \frac{|y|^2 + |x|^2 - 2xy}{|x-y|^n} - \frac{2x(x-y)}{|x-y|^n} \\ &= \frac{1}{|x-y|^{n-2}} - \frac{2x_i(x_i - y_i)}{|x-y|^n}. \end{aligned}$$

The last term is harmonic since it is the derivative with respect to y_i of $|x-y|^{2-n}$. Thus, $v(x,y)$ is harmonic in both x and y variables.

It would be difficult to verify the above aforementioned functions solved Laplace's equation, but they are readily constructed from simpler solutions.

Now, let us consider $|y| = R$ and define

$$\tilde{v}(x) = \int_{|y|=R} V(x,y)\, dS(y).$$

First, it is clear that $\tilde{v}(x)$ is harmonic for $x \in B_R(0)$. Indeed for any fixed x in the $B_R(0)$, $V(x,y)$ is bounded. So we may interchange the Laplacian with the integral to get that $\tilde{v}(x)$ is harmonic in $B_R(0)$. Next, we claim that $\tilde{v}(x)$ is radial, i.e. only depends on $|x|$. To show this we consider an arbitrary rotation transformation on $\tilde{v}(x)$. Sometimes this is called an orthonormal transformation as it corresponds to the transformation of $x = Px'$, where P is an orthonormal matrix (rows/columns are orthogonal basis in $\mathbb{R}^n$, $P^T = P^{-1}$ is another property). So, given the transformation, we have

$$\begin{aligned}\tilde{v}(Px) &= \int_{|y|=R} \frac{|y|^2 - |z|^2}{|Px - y|^n}\, dS(y) \\ &= \int_{|y|=R} \frac{R^2 - |z|^2}{|Px - y|^n}\, dS(y).\end{aligned}$$

Now, we change variables with $Pz = y$:

$$\begin{aligned}\tilde{v}(Px) &= \int_{|Pz|=R} \frac{R^2 - |Px|^2}{|Px - Pz|^n}\, dS(Pz) \\ &= \int_{|z|=R} \frac{R^2 - |x|^2}{|P(x-z)|^n} \det(P)\, dS(z) \\ &= \int_{|z|=R} \frac{R^2 - |x|^2}{|x - z|^n}\, dS(z).\end{aligned}$$

In the above, we have used the fact that rotations obviously do not change vector magnitude. Also, we know $\det(P) = 1$ as $1 = \det(I) = \det(PP^{-1}) = \det(P)\det(P^{-1}) = \det(P)\det(P^T) = \det(P)^2$. So, $\tilde{v}(x)$ is rotationally invariant. Thus, we calculate

$$\tilde{v}(0) = \int_{|y|=R} R^{2-n}\, dS(y) = R^{2-n} A(S_R(0)) = n\omega_n R.$$

It turns out the above result is valid for any $x \in B_R(0)$, this can be verified by carrying out the integration; but this is rather complicated.

Now, we can make the following definition

Definition 5.1. *The* Poisson Kernel*:*

$$K(x,y) = \frac{R^2 - |x|^2}{n\omega_n R} \frac{1}{|x-y|^n},$$

where $|y| = R$*.*

From the above, we already know that $K(x,y)$ is harmonic in $B_R(0)$ with respect to x. From the calculation for $\tilde{v}(x)$, we also have

$$\int_{|y|=R} K(x,y)\, dS(y) = 1$$

by construction. Next, we have

Definition 5.2. *The* Poisson Integral *for* $x \in B_R(0)$ *is*

$$w(x) = \int_{|y|=R} K(x,y)\phi(y)\, dS(y).$$

Theorem 5.3. *(Properties of the Poisson Integral) Given the above definition,* $w \in C^\infty(B_R(0)) \cap C^0(\overline{B_R(0)})$ *with* $\Delta w = 0$ *in* $B_R(0)$*. Also* $w(x) \to \phi(y)$ *as* $x \to y \in \partial B_R(0)$*.*

Remark: Basically one has that w solves the Dirichlet problem for Laplace's equation:

$$\begin{aligned} \Delta w = 0 \quad &\text{in} \quad B_R(0) \\ u = \phi \quad &\text{on} \quad \partial B_R(0) \end{aligned}$$

Proof: $w \in C^\infty(B_R(0))$ is obvious since for fixed $x \in B_R(0)$, $K(x,y)$ is bounded and harmonic. Hence on may take derivatives of any order inside the integral (the derivatives will also be well behaved as $x \neq y$ for our fixed x). So, all we need to show is that $w(x) \to \phi(y_0)$ as $x \to y_0 \in \partial B_R(0)$. Take an arbitrary $\epsilon > 0$ and choose δ such that $|\phi(y) - \phi(y_0)| < \epsilon$ implies

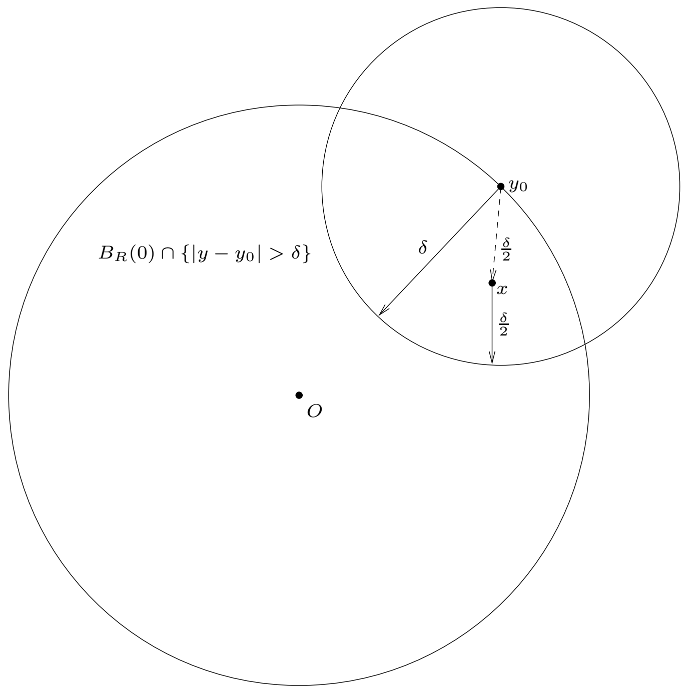

Figure 5.2:

$|y - y_0| < \delta$. Keeping in mind the figure, we calculate:

$$
\begin{aligned}
|w(x) - \phi(y_0)| &= \left| \int_{\partial B_R(0)} K(x,y)(\phi(y) - \phi(y_0))\, dS(y) \right| \\
&\leq \int_{\partial B_R(0) \cap \{|y-y_0|<\delta\}} |K(x,y)(\phi(y) - \phi(y_0))|\, dS(y) \\
&\qquad + \int_{\partial B_R(0) \cap \{|y-y_0|>\delta\}} |K(x,y)(\phi(y) - \phi(y_0))|\, dS(y) \\
&\leq \epsilon \int_{\partial B_R(0) \cap \{|y-y_0|<\delta\}} |K(x,y)|\, dS(y) \\
&\qquad + 2 \cdot \max_{\partial B_R(0)} \phi \cdot \int_{\partial B_R(0) \cap \{|y-y_0|>\delta\}} \frac{R^2 - |x|^2}{|x-y|^n}\, dS(y).
\end{aligned}
$$

Now, if one chooses x such that $|x - y_0| \leq \frac{\delta}{2}$, this implies that $|x - y| \geq \frac{\delta}{2}$ for $y \in \partial B_R(0) \cap \{|y - y_0| > \delta\}$. Thus, we have

$$
|w(x) - \phi(y_0)| \leq \epsilon + 2 \cdot \max_{\partial B_R(0)} \phi \cdot \frac{R^2 - |x|^2}{\frac{\delta}{2}^n}
$$

$$\leq \quad 2\epsilon.$$

The last inequality comes from simply picking x close enough to y_0 (as this happens $R^2-|x|^2$ clearly shrinks). So, we have now shown that $w(x) \to \phi(y_0)$ as $x \to y_0 \in \partial B_R(0)$. This also implies the continuity of $w(x)$ on the closure of Ω. ■

A very important consequence to the las theorem is that if $u \in C^2(\Omega)$ is harmonic, then for any ball $B_R(z)$, one has

$$u(x) = \frac{R^2 - |x-z|^2}{n\omega_n R} \int_{\partial B_R(z)} \frac{u(y)}{|x-y|^n}\, dS(y).$$

In other words, the value of a harmonic function at any point is completely determined by the values it takes on any given spherical shell surrounding that point! Taking, $x = z$ in the above, we get the following:

Theorem 5.4. *(Mean Value Property) If $u \in C^2(\Omega)$ and is harmonic on Ω, then one has*

$$\begin{aligned} u(z) &= \frac{1}{n\omega_n R^{n-1}} \int_{\partial B_R(z)} u(y)\, dS(y) \\ &= \frac{1}{A(\partial B_R(z))} \int_{\partial B_R(z)} u(y)\, dS(y), \end{aligned} \tag{7}$$

for any $z \in \Omega$, $B_R(z) \subset \Omega$.

In addition, one has

Corollary 5.5. *If $u \in C^2(\Omega)$ and is subharmonic(superharmonic) on Ω, then one has*

$$u(z) \leq (\geq) \frac{1}{A(\partial B_R(z))} \int_{\partial B_R(z)} u(y)\, dS(y), \tag{8}$$

for any $z \in \Omega$, $B_R(z) \subset \Omega$.

Proof: This is a simple matter of solving the following Dirichlet Problem:

$$\begin{cases} \Delta v = 0 & \text{in} \quad B_R(z) \\ v = u & \text{on} \quad \partial B_R(z) \end{cases}$$

First, we know that $\Delta(u - v) \geq (\leq)\, 0$. Thus, the weak maximum principle states that $u \leq (\geq)\, v$ in $\partial B_R(z)$. So, putting everything together, one has

$$\begin{aligned} u \leq (\geq)\, v &= \frac{1}{A(\partial B_R(z))} \int_{\partial B_R(z)} v(y)\, dS(y) \\ &= \frac{1}{A(\partial B_R(z))} \int_{\partial B_R(z)} u(y)\, dS(y). \end{aligned}$$ ■

6 Strong Maximum Principle

Theorem 6.1 (Strong Maximum Principle). *If $u \in C^2(\Omega)$ with Ω connected and $\Delta u \geq (\leq)\, 0$, then u can not take on interior maximum(minimum) in Ω, unless u is constant.*

Proof: Suppose $u(y) = M := \max_\Omega u$. Clearly, one has

$$\Delta(M - u) \leq (\geq)\, 0.$$

Thus, by the corollary to the mean value property,

$$(M-u)(y) \geq (\leq)\, \frac{1}{n\omega_n R} \int_{\partial B_R(y)} M - u(x)\, dS(x)$$

for any $B_R(y) \subset \Omega$. This implies that $u = M$ on $B_R(y)$, which in turn implies $u = M$ on Ω as Ω is connected. ■

Problem: Take $\Omega \subset \mathbb{R}^n$ bounded and $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$. If

$$\begin{aligned} \Delta u = u^3 - u \quad &\text{in} \quad \Omega \\ u = 0 \quad &\text{on} \quad \partial\Omega \end{aligned}$$

Prove $-1 \leq u \leq 1$. Can u take either value ± 1?

7 Estimates for Derivatives of Harmonic Functions

Now, we will go through estimating the derivatives of harmonic functions. First, note that the mean value property immediately implies that

$$u(x) = \frac{1}{\omega_n R^n} \int_{B_R(x)} u(y)\, dy$$

for any $B_R(x) \subset \Omega$ and u harmonic. This can simply be ascertained from preforming a radial integration on the statement of the mean value property. As we have shown derivatives to be harmonic, we have that

$$D_i u(x) = \frac{1}{\omega_n R^n} \int_{B_R(x)} \operatorname{div} u^{,i}\, dy,$$

where w^i is a vector whose ith place is u and whose other components are zero. So applying the divergence theorem to the above, we have

$$\begin{aligned} D_i u(x) &= \frac{1}{\omega_n R^n} \int_{\partial\partial B_R(x)} w^i \cdot \nu \, dS(y) \\ &\leq \frac{1}{\omega_n R^n} \sup_{B_R(x)} |u| \cdot \int_{\partial\partial B_R(x)} dS(y) \\ &= \frac{n}{R} \sup_{\partial B_R(x)} |u|. \end{aligned}$$

From this we see that any derivative is bounded by the final term in the above, thus

$$|Du(x)| \leq \frac{n}{R} \sup_{B_R(x)} |u|.$$

Applying this result iteratively over concentric balls whose radius increases

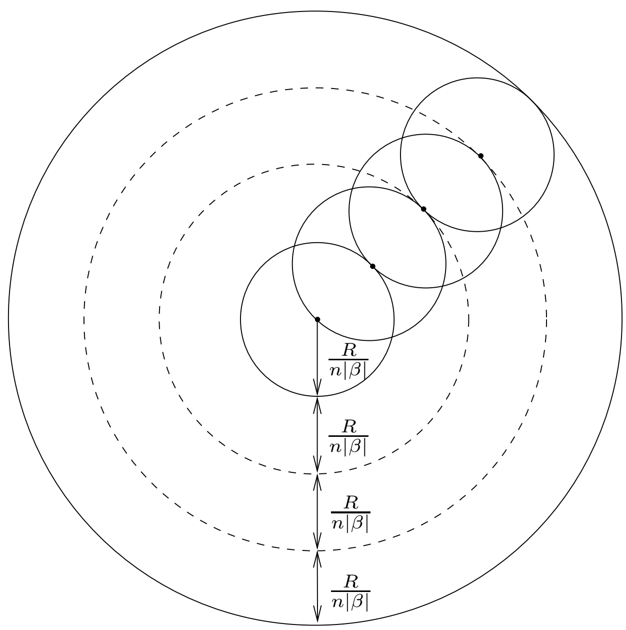

Figure 7.3:

by $R/|\beta|$ for each iteration yields the following (refer to the figure):

$$|D^\beta u(x)| \leq \left(\frac{n|\beta|}{R}\right)^{|\beta|} \sup_{B_R(x)} |u|.$$

From this equation, the following is clear.

Theorem 7.1. *Consider Ω bounded and $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$. If u is harmonic in Ω, then for any $\Omega' \Subset \Omega$ the following estimate holds*

$$|D^\beta u(x)| \leq \left(\frac{n|\beta|}{d_{\Omega'}}\right)^{|\beta|} \sup_{\overline{\Omega}} |u| \quad \forall x \in \Omega', \tag{9}$$

where $d_{\Omega'} = dist(\Omega', \partial\Omega)$.

8 Convergence Theorems

Important to the proof of existence of solutions to Laplace's equation is the concept of convergence of harmonic functions.

Theorem 8.1. *A bounded sequence of harmonic functions on a domain Ω contains a subsequence which converges uniformly, together with its derivatives, to a harmonic function.*

Proof: Take $\Omega' \subset \Omega$ compact. Now, we calculate via the mean value theorem of calculus:

$$\begin{aligned} |u_m(x) - u_n(x)| &\leq \max_{\substack{tx+(1-t)y \\ 0 \leq t \leq 1}} |Du_m| \cdot |x-y| \\ &\leq \max_{\Omega'} |Du_m| \cdot |x-y| \\ &\leq C \sup_{\Omega'} |u_m| \cdot |x-y| \\ &\leq C'(K, n, L)|x-y|, \end{aligned}$$

where L is a constant which bounds the sequence elements u_m. From this, we see that the sequence is equicontinuous. Next we recall that the Azalea-Ascoli theorem asserts that such a sequence will have a subsequence converging uniformly to a limit. Now, one may apply the same calculation above iteratively to any order of derivative (iteratively refining the subsequence for each new derivative). From this we conclude that there is a subsequence whose derivatives up to and including order $|\beta|$ all converge uniformly to a limit for any $|\beta|$. It now follows that the limit is indeed a solution to Laplace's equation. ■

9 Solution of the Dirichlet Problem by the Perron Process

Before proving the existence of solution to the Dirichlet problem for Laplace's equation, one needs to extend the notions of subharmonic(superharmonic) functions to non-C^2 functions.

Definition 9.1. *u is* subharmonic(superharmonic) *in Ω if for any ball $B \Subset \Omega$ and harmonic function $h \in C^2(B) \cap C^0(\overline{B})$ with $h \geq (\leq)\, u$ on ∂B, one has $h \geq (\leq)\, u$ on B.*

Note: If u is subharmonic and superharmonic, then it is harmonic.

From this new definition, some properties are immediate.
Properties:

i.) The Strong Maximum Principle still applies to this extended definition.

ii.) $u = \max u_1, \ldots, u_m$ is subharmonic if $u_1, \ldots, u_m$ are subharmonic.

iii.) Harmonic lifting.