Lectures on Partial Differential Equations:Day 3

January 14, 2005

1 Strong Maximum Principle

Theorem 1.1 (Strong Maximum Principle). *If $u \in C^2(\Omega)$ with Ω connected and $\Delta u \geq (\leq)\, 0$, then u can not take on interior maximum(minimum) in Ω, unless u is constant.*

Proof: Suppose $u(y) = M := \max_\Omega u$. Clearly, one has

$$\Delta(M - u) \leq (\geq)\, 0.$$

Thus, by the corollary to the mean value property,

$$(M-u)(y) \geq (\leq)\, \frac{1}{n\omega_n R} \int_{\partial B_R(y)} M - u(x)\, dS(x)$$

for any $B_R(y) \subset \Omega$. This implies that $u = M$ on $B_R(y)$, which in turn implies $u = M$ on Ω as Ω is connected. ■

Problem: Take $\Omega \subset \mathbb{R}^n$ bounded and $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$. If

$$\begin{aligned} \Delta u = u^3 - u \quad &\text{in} \quad \Omega \\ u = 0 \quad &\text{on} \quad \partial\Omega \end{aligned}$$

Prove $-1 \leq u \leq 1$. Can u take either value ± 1?

2 Estimates for Derivatives of Harmonic Functions

Now, we will go through estimating the derivatives of harmonic functions. First, note that the mean value property immediately implies that

$$u(x) = \frac{1}{\omega_n R^n} \int_{B_R(x)} u(y)\, dy$$

for any $B_R(x) \subset \Omega$ and u harmonic. This can simply be ascertained from preforming a radial integration on the statement of the mean value property. As we have shown derivatives to be harmonic, we have that

$$D_i u(x) = \frac{1}{\omega_n R^n} \int_{B_R(x)} \operatorname{div} u^{,i} \, dy,$$

where $u^{,i}$ is a vector whose ith place is u and whose other components are zero. So applying the divergence theorem to the above, we have

$$\begin{aligned} D_i u(x) &= \frac{1}{\omega_n R^n} \int_{\partial\partial B_R(x)} u^{,i} \cdot \nu \, dS(y) \\ &\leq \frac{1}{\omega_n R^n} \sup_{B_R(x)} |u| \cdot \int_{\partial\partial B_R(x)} dS(y) \\ &= \frac{n}{R} \sup_{\partial B_R(x)} |u|. \end{aligned}$$

From this we see that any derivative is bounded by the final term in the above, thus

$$|Du(x)| \leq \frac{n}{R} \sup_{B_R(x)} |u|.$$

Applying this result iteratively over concentric balls whose radius increases by $R/|\beta|$ for each iteration yields the following (refer to the figure):

$$|D^\beta u(x)| \leq \left(\frac{n|\beta|}{R}\right)^{|\beta|} \sup_{B_R(x)} |u|.$$

From this equation, the following is clear.

Theorem 2.1. *Consider Ω bounded and $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$. If u is harmonic in Ω, then for any $\Omega' \Subset \Omega$ the following estimate holds*

$$|D^\beta u(x)| \leq \left(\frac{n|\beta|}{d_{\Omega'}}\right)^{|\beta|} \sup_{\overline{\Omega}} |u| \quad \forall x \in \Omega', \tag{1}$$

where $d_{\Omega'} = dist(\Omega', \partial\Omega)$.

3 Convergence Theorems

Important to the proof of existence of solutions to Laplace's equation is the concept of convergence of harmonic functions.

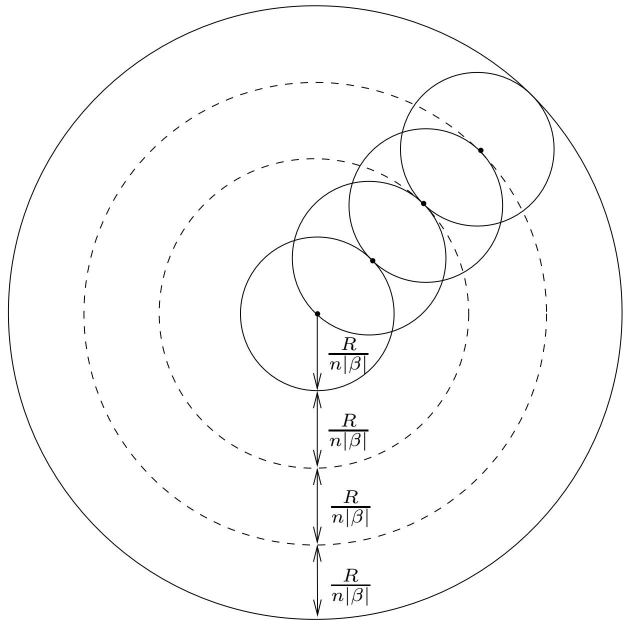

Figure 2.1:

Theorem 3.1. *A bounded sequence of harmonic functions on a domain Ω contains a subsequence which converges uniformly, together with its derivatives, to a harmonic function.*

Proof: Take $\Omega' \subset \Omega$ compact. Now, we calculate via the mean value theorem of calculus:

$$\begin{aligned} |u_m(x) - u_n(x)| &\leq \max_{\substack{tx + (1-t)y \\ 0 \leq t \leq 1}} |Du_m| \cdot |x - y| \\ &\leq \max_{\Omega'} |Du_m| \cdot |x - y| \\ &\leq C \sup_{\Omega'} |u_m| \cdot |x - y| \\ &\leq C'(K, n, L)|x - y|, \end{aligned}$$

where L is a constant which bounds the sequence elements u_m. From this, we see that the sequence is equicontinuous. Next we recall that the Azalea-Ascoli theorem asserts that such a sequence will have a subsequence converging uniformly to a limit. Now, one may apply the same calculation above iteratively to any order of derivative (iteratively refining the subsequence for each new derivative). From this we conclude that there is a subsequence whose derivatives up to and including order $|\beta|$ all converge uniformly to a limit for any $|\beta|$. It now follows that the limit is indeed a solution to Laplace's equation. ■

4 Solution of the Dirichlet Problem by the Perron Process

Before proving the existence of solution to the Dirichlet problem for Laplace's equation, one needs to extend the notions of subharmonic(superharmonic) functions to non-C^2 functions.

Definition 4.1. *u is* subharmonic(superharmonic) *in Ω if for any ball $B \Subset \Omega$ and harmonic function $h \in C^2(B) \cap C^0(\overline{B})$ with $h \geq (\leq)\, u$ on ∂B, one has $h \geq (\leq)\, u$ on B.*

Note: If u is subharmonic and superharmonic, then it is harmonic.

From this new definition, some properties are immediate.
Properties:

i.) The Strong Maximum Principle still applies to this extended definition.

ii.) $u = \max u_1, \ldots, u_m$ is subharmonic if $u_1, \ldots, u_m$ are subharmonic.

iii.) Harmonic lifting.