Lectures on Partial Differential Equations:Day 3

January 13, 2005

0.1 Weak Maximum Principle for General Elliptic Equations

Now, we will go over the weak maximum principle for operators having all order of derivatives ≤ 2.

Theorem 0.1 (Weak Maximum Principle). *Consider $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$; $a^{ij}, b^i, c : \Omega \to \mathbb{R}$; $c \leq 0$; and Ω bounded. If u satisfies the elliptic equation (i.e. $[a^{ij}] = \mathcal{A} > 0$)*

$$Lu = a^{ij} D_{ij} + b^i D_i u + cu \geq f, \tag{1}$$

then

$$u \leq \max_{\partial\Omega} u^+ + C\left(n, \operatorname{diam}\Omega, \sup_{\overline{\Omega}} \frac{|b|}{\lambda}\right) \cdot \sup_{\overline{\Omega}} \frac{|f|}{\lambda}, \tag{2}$$

where

$$\lambda = \min_{|\xi|=1} a^{ij}\xi_i\xi_j \quad \textit{and} \quad u^+ = \max\{u, 0\}.$$

Note: If a^{ij} is not constant, then λ will almost always be non-constant on Ω as well.

Consequences:

- If $f = 0$, then $u \leq \max_{\partial\Omega} u^+$.
- If $f = c = 0$, then $u \leq \max_{\partial\Omega} u^+$.
- If $Lu \leq f$ one gains a lower bound

  $$u \geq \max_{\partial\Omega} u^- - C \sup_{\overline{\Omega}} \frac{|f|}{\lambda},$$

  where $u^- = \min\{u, 0\}$.

- If we have the PDE $Lu = f$, then one has both an upper and lower bound:

$$|u| \leq \max_{\partial\Omega} |u| + C \sup_{\overline{\Omega}} \frac{|f|}{\lambda}.$$

- Uniqueness of the Dirichlet Problem. Recall, that the problem is as follows

$$\begin{cases} \text{PDE:} \quad Lu = f & \text{in} \quad \Omega \\ \qquad\quad\; u = \phi & \text{on} \quad \partial\Omega \end{cases}.$$

  So if one has $u, v \in C^2(\Omega) \cap C^0(\overline{\Omega})$ with $Lu = Lv$ in Ω, and $u = v$ on $\partial\Omega$, then $u \equiv v$ in Ω. The proof is the simple application of the weak maximum principle to $w = u - v$, which is obviously a solution of PDE as it's linear.

Proof of Weak Max. Principle: As in previous proofs, we analyze a particular auxiliary function to prove the result.

Aside: $v = |x|^2$ will not work in this situation as $Lv = 2 \cdot \operatorname{Tr} \mathcal{A} + 2b^i x_i + c|x|^2$; the $b^i x_i$ may be a large negative number for large values of $|x|$.

So, let us try

$$v = e^{\alpha(x,\xi)},$$

where ξ is some arbitrary vector. Also, we will consider $\Omega^+ = \{x \in \Omega \,|\, u > 0\}$ instead of Ω, for the rest of the proof. Clearly, this does not affect the result as we are seeking to bound u by $\max_{\partial\Omega} u^+$. With that, we have on Ω^+

$$\begin{aligned} L_0 u := a^{ij} D_{ij} u + b^i D_i u &\geq -cu + f \\ &\geq f. \end{aligned}$$

Now, we wish to specify which vector ξ is. For the following, please refer to the figure. We first choose our coordinates so that the origin corresponds to a boundary point of Ω such that there exists a hyperplane through that point with Ω lying on one side of it. ξ is simply taken to be the unit vector normal to this hyperplane. For simplicity we rotate our coordinate system such that $\xi_i = 0$ for $i > 0$. Next, we calculate

$$\begin{aligned} L_0 v &= (\alpha^2 a^{ij} \xi_i \xi_j + \alpha b^i \xi_i) e^{\alpha(x,\xi)} \\ &= (\alpha^2 a^{ij} \xi_i \xi_j + \alpha b^i \xi_i) e^{\alpha(x,\xi)} \\ &\geq \alpha \lambda e^{\alpha(x,\xi)}, \end{aligned}$$

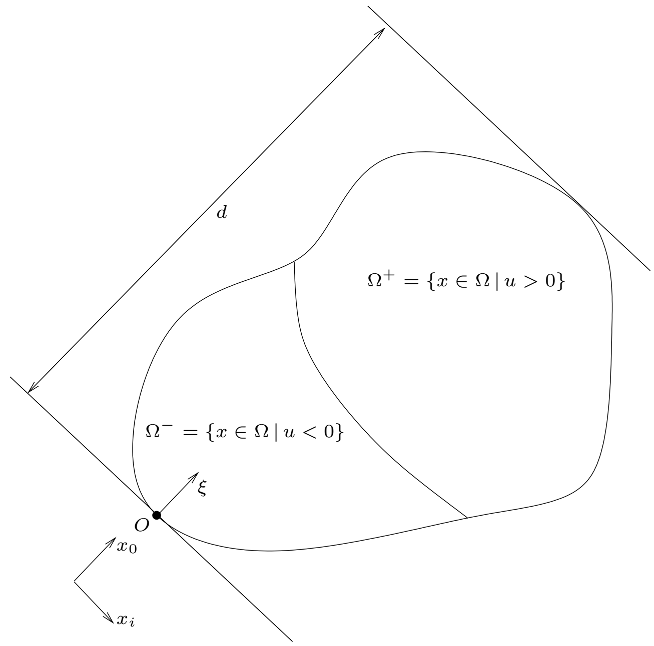

Figure 0.1:

provided we choose $\alpha \geq \left\{ sup_{\overline{Omega}} \frac{|b|}{\lambda} + 1 \right)$.

Aside: The calculation for the determination of α is as follows:

$$\alpha^2 a^{ij} \xi_i \xi_j + \alpha b^i \xi_i \quad \geq \quad \alpha^2 \lambda + \alpha b^i \xi_i.$$

So, $\alpha^2 \lambda + \alpha b^i \xi_i \geq \alpha \lambda$ implies that

$$\begin{aligned} \alpha \lambda &\geq \lambda - b^i \xi_i \\ \Longrightarrow \alpha &\geq 1 - \frac{1}{\lambda} \sup_{\overline{\Omega}} |b| \\ \Longrightarrow \alpha &\geq 1 - \sup_{\overline{\Omega}} \frac{|b|}{\lambda}. \end{aligned}$$

For convenience, define $\tilde{b} = \sup_{\overline{\Omega}} \frac{|b|}{\lambda}$. So, clearly we have the desired inequality if we choose $\alpha \geq \tilde{b} + 1$.

Given our choice of ξ, it is clear that $(x, \xi) \geq 0$ for all $x \in \Omega$. Thus, one sees that $L_0 v \geq \alpha \lambda$. So, one can ascertain that

$$L_0(u^+ + kv) \geq f + \alpha \lambda k > 0 \text{ in } \Omega^+,$$

provided k is chosen so that

$$k > \frac{1}{\alpha} \sup_{\overline{\Omega}} \frac{|f|}{\lambda}.$$

Now, the proof proceeds as previous ones. Suppose $w := u^+ + kv$ has a positive maximum in Ω^+ at a point y. In this situation we have

$$[D_i j w(y)] \leq 0 \Longrightarrow a^{ij} D_{ij} w(y) \leq 0, \; b^i D_i w(y) = 0,$$

which implies $Lw \leq 0$; a contradiction. So, this implies the second inequality of the following:

$$\begin{aligned} u + \frac{e^0}{\alpha} \sup_{\overline{\Omega}} \frac{|f|}{\lambda} &\leq w \\ &\leq \max_{\partial\Omega^+} w \\ &\leq \max_{\partial\Omega^+} u^+ + \frac{e^{\alpha d}}{\alpha} \cdot \sup_{\overline{\Omega}} \frac{|f|}{\lambda} \\ &\leq \max_{\partial\Omega} u^+ + \frac{e^{\alpha d}}{\alpha} \cdot \sup_{\overline{\Omega}} \frac{|f|}{\lambda}, \end{aligned}$$

again where $\alpha = b_0 + 1$ and d is the breadth of Ω in the direction of ξ. From the above, we finally get

$$u \leq \max_{\partial\Omega} u^+ + \frac{e^{\alpha d} - 1}{\alpha} \cdot \sup_{\overline{\Omega}} \frac{|f|}{\lambda}. \qquad \blacksquare$$

Remark: Given the above, one needs $\frac{|b|}{\lambda}$ and $\frac{|f|}{\lambda}$ bounded for the weak maximum principle to give a non-trivial result. Then one may apply the weak maximum principle to get uniqueness.

1 Poisson Integral Formula

The best possible scenario one can hope for when dealing in theoretical PDE is to actually have an analytic solution to the equation. Obviously, it is easier to analyze a solution directly rather than having to do so through its properties. Fortunately, this turns out to be possible for the Dirichlet problem for Laplace's equation in a ball.

To start off, let us consider some other examples of harmonic functions:

- Since Laplace's equation is a constant coefficient equation, we know that if $u \in C^3(\Omega)$ is harmonic, then so is all its partial derivatives. Indeed,

$$D_i(\Delta u) = 0 \iff \Delta(D_i u) = 0.$$

- There are also radially symmetric solutions to Laplace's equation. To derive these, let $r = |x|$ and $u(x) = g(r)$. We calculate

$$\begin{aligned} D_i u &= g' \cdot \frac{x_i}{|x|} = \frac{g'}{r} \cdot x_i \\ \Delta u &= D_{ii} u = n \cdot \frac{g'}{r} + \frac{g''}{r^2} \cdot x_i^2 - \frac{g'}{r^3} x_i^2 \\ &= n \cdot \frac{g'}{r} + g'' - \frac{g'}{r} \\ &= (n-1) \cdot \frac{g'}{r} + g'' \\ &= \frac{(r^{n-1} g')'}{r^{n-1}} = 0. \end{aligned}$$

This implies that $r^{n-1} g'(r) = \text{constant}$; integration then yields

$$g = \begin{cases} C_1 r^{2-n} + C_2 & n > 2 \\ C_1 \ln r + C_2 & n = 2 \end{cases}.$$

These are all the radial solution of the homogeneous Laplace's equation outside of the origin. For simplicity we will take $C_1 = 1$ and $C_2 = 0$ in the above.

- It is easily verified that translating the above solutions so that y is taken to be the new origin, are also harmonic functions:

$$\begin{cases} |x-y|^{2-n} & n > 2 \\ C_1 \ln |x-y| & n = 2 \end{cases}.$$

Since these functions are symmetric in x and y, it's obvious the above are harmonic with respect to either x or y provided $x \neq y$. Taking a derivative of the above indicates that

$$\frac{x_i - y_i}{|x_i - y_i|^n}$$

is also harmonic. Next consider

$$\begin{aligned} V(x,y) &= \frac{|y|^2-|x|^2}{|x-y|^n} = \frac{|y|^2+|x|^2-2xy}{|x-y|^n} - \frac{2x(x-y)}{|x-y|^n} \\ &= \frac{1}{|x-y|^{n-2}} - \frac{2x_i(x_i-y_i)}{|x-y|^n}. \end{aligned}$$

The last term is harmonic since it is the derivative with respect to y_i of $|x-y|^{2-n}$. Thus, $v(x,y)$ is harmonic in both x and y variables.

It would be difficult to verify the above aforementioned functions solved Laplace's equation, but they are readily constructed from simpler solutions.

Now, let us consider $|y| = R$ and define

$$\tilde{v}(x) = \int_{|y|=R} V(x,y)\, dS(y).$$

First, it is clear that $\tilde{v}(x)$ is harmonic for $x \in B_R(0)$. Indeed for any fixed x in the $B_R(0)$, $V(x,y)$ is bounded. So we may interchange the Laplacian with the integral to get that $\tilde{v}(x)$ is harmonic in $B_R(0)$. Next, we claim that $\tilde{v}(x)$ is radial, i.e. only depends on $|x|$. To show this we consider an arbitrary rotation transformation on $\tilde{v}(x)$. Sometimes this is called an orthonormal transformation as it corresponds to the transformation of $x = Px'$, where P is an orthonormal matrix (rows/columns are orthogonal basis in $\mathbb{R}^n$, $P^T = P^{-1}$ is another property). So, given the transformation, we have

$$\begin{aligned} \tilde{v}(Px) &= \int_{|y|=R} \frac{|y|^2-|z|^2}{|Px-y|^n}\, dS(y) \\ &= \int_{|y|=R} \frac{R^2-|z|^2}{|Px-y|^n}\, dS(y). \end{aligned}$$

Now, we change variables with $Pz = y$:

$$\begin{aligned} \tilde{v}(Px) &= \int_{|Pz|=R} \frac{R^2-|Px|^2}{|Px-Pz|^n}\, dS(Pz) \\ &= \int_{|z|=R} \frac{R^2-|x|^2}{|P(x-z)|^n} \det(P)\, dS(z) \\ &= \int_{|z|=R} \frac{R^2-|x|^2}{|x-z|^n}\, dS(z). \end{aligned}$$

In the above, we have used the fact that rotations obviously do not change vector magnitude. Also, we know $\det(P) = 1$ as $1 = \det(I) = \det(PP^{-1}) = \det(P)\det(P^{-1}) = \det(P)\det(P^T) = \det(P)^2$. So, $\tilde{v}(x)$ is rotationally invariant. Thus, we calculate

$$\tilde{v}(0) = \int_{|y|=R} R^{2-n}\, dS(y) = R^{2-n} A(S_R(0)) = n\omega_n R.$$

It turns out the above result is valid for any $x \in B_R(0)$, this can be verified by carrying out the integration; but this is rather complicated.

Now, we can make the following definition

Definition 1.1. *The* Poisson Kernel*:*

$$K(x,y) = \frac{R^2 - |x|^2}{n\omega_n R} \frac{1}{|x-y|^n},$$

where $|y| = R$.

From the above, we already know that $K(x,y)$ is harmonic in $B_R(0)$ with respect to x. From the calculation for $\tilde{v}(x)$, we also have

$$\int_{|y|=R} K(x,y)\, dS(y) = 1$$

by construction. Next, we have

Definition 1.2. *The* Poisson Integral *for* $x \in B_R(0)$ *is*

$$w(x) = \int_{|y|=R} K(x,y)\phi(y)\, dS(y).$$

Theorem 1.3. *(Properties of the Poisson Integral) Given the above definition,* $w \in C^\infty(B_R(0)) \cap C^0(\overline{B_R(0)})$ *with* $\Delta w = 0$ *in* $B_R(0)$*. Also* $w(x) \to \phi(y)$ *as* $x \to y \in \partial B_R(0)$.

Remark: Basically one has that w solves the Dirichlet problem for Laplace's equation:

$$\begin{aligned} \Delta w = 0 \quad &\text{in} \quad B_R(0) \\ u = \phi \quad &\text{on} \quad \partial B_R(0) \end{aligned}$$

Proof: $w \in C^\infty(B_R(0))$ is obvious since for fixed $x \in B_R(0)$, $K(x,y)$ is bounded and harmonic. Hence on may take derivatives of any order inside

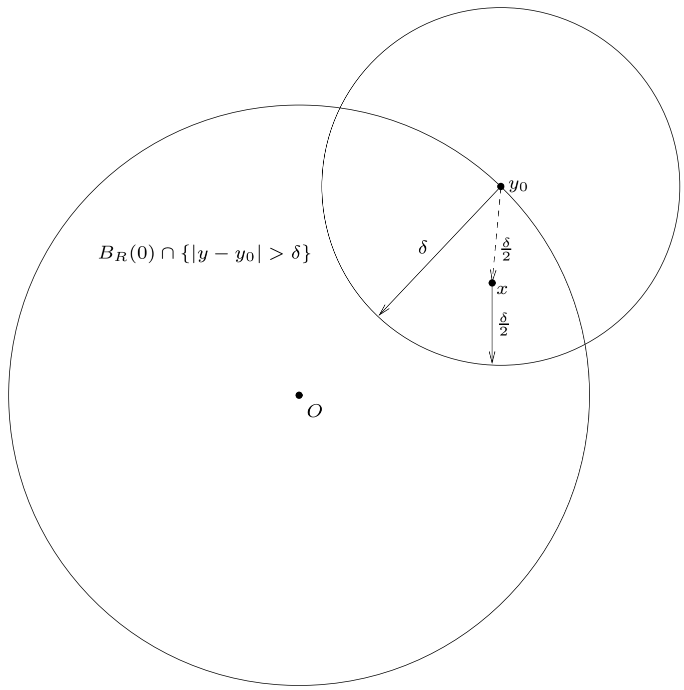

Figure 1.2:

the integral (the derivatives will also be well behaved as $x \neq y$ for our fixed x). So, all we need to show is that $w(x) \to \phi(y_0)$ as $x \to y_0 \in \partial B_R(0)$. Take an arbitrary $\epsilon > 0$ and choose δ such that $|\phi(y) - \phi(y_0)| < \epsilon$ implies $|y - y_0| < \delta$. Keeping in mind the figure, we calculate:

$$\begin{aligned}
|w(x) - \phi(y_0)| &= \left| \int_{\partial B_R(0)} K(x,y)(\phi(y) - \phi(y_0))\, dS(y) \right| \\
&\leq \int_{\partial B_R(0) \cap \{|y-y_0|<\delta\}} |K(x,y)(\phi(y) - \phi(y_0))|\, dS(y) \\
&\qquad + \int_{\partial B_R(0) \cap \{|y-y_0|>\delta\}} |K(x,y)(\phi(y) - \phi(y_0))|\, dS(y) \\
&\leq \epsilon \int_{\partial B_R(0) \cap \{|y-y_0|<\delta\}} |K(x,y)|\, dS(y) \\
&\qquad + 2 \cdot \max_{\partial B_R(0)} \phi \cdot \int_{\partial B_R(0) \cap \{|y-y_0|>\delta\}} \frac{R^2 - |x|^2}{|x-y|^n}\, dS(y).
\end{aligned}$$

Now, if one chooses x such that $|x - y_0| \leq \frac{\delta}{2}$, this implies that $|x - y| \geq \frac{\delta}{2}$ for $y \in \partial B_R(0) \cap \{|y - y_0| > \delta\}$. Thus, we have

$$
\begin{aligned}
|w(x) - \phi(y_0)| &\leq \epsilon + 2 \cdot \max_{\partial B_R(0)} \phi \cdot \frac{R^2 - |x|^2}{\frac{\delta}{2}^n} \\
&\leq 2\epsilon.
\end{aligned}
$$

The last inequality comes from simply picking x close enough to y_0 (as this happens $R^2 - |x|^2$ clearly shrinks). So, we have now shown that $w(x) \to \phi(y_0)$ as $x \to y_0 \in \partial B_R(0)$. This also implies the continuity of $w(x)$ on the closure of Ω. ■

A very important consequence to the las theorem is that if $u \in C^2(\Omega)$ is harmonic, then for any ball $B_R(z)$, one has

$$
u(x) = \frac{R^2 - |x - z|^2}{n\omega_n R} \int_{\partial B_R(z)} \frac{u(y)}{|x - y|^n} \, dS(y).
$$

In other words, the value of a harmonic function at any point is completely determined by the values it takes on any given spherical shell surrounding that point! Taking, $x = z$ in the above, we get the following:

Theorem 1.4. *(Mean Value Property) If $u \in C^2(\Omega)$ and is harmonic on Ω, then one has*

$$
\begin{aligned}
u(z) &= \frac{1}{n\omega_n R^{n-1}} \int_{\partial B_R(z)} u(y) \, dS(y) \\
&= \frac{1}{A(\partial B_R(z))} \int_{\partial B_R(z)} u(y) \, dS(y), \qquad (3)
\end{aligned}
$$

for any $z \in \Omega$, $B_R(z) \subset \Omega$.

In addition, one has

Corollary 1.5. *If $u \in C^2(\Omega)$ and is subharmonic(superharmonic) on Ω, then one has*

$$
u(z) \leq (\geq) \frac{1}{A(\partial B_R(z))} \int_{\partial B_R(z)} u(y) \, dS(y), \qquad (4)
$$

for any $z \in \Omega$, $B_R(z) \subset \Omega$.

Proof: This is a simple matter of solving the following Dirichlet Problem:

$$\begin{cases} \Delta v = 0 & \text{in} \quad B_R(z) \\ v = u & \text{on} \quad \partial B_R(z) \end{cases}$$

First, we know that $\Delta(u - v) \geq (\leq)\, 0$. Thus, the weak maximum principle states that $u \leq (\geq)\, v$ in $\partial B_R(z)$. So, putting everything together, one has

$$\begin{aligned} u \leq (\geq)\, v &= \frac{1}{A(\partial B_R(z))} \int_{\partial B_R(z)} v(y)\, dS(y) \\ &= \frac{1}{A(\partial B_R(z))} \int_{\partial B_R(z)} u(y)\, dS(y). \qquad \blacksquare \end{aligned}$$