Lectures on Partial Differential Equations:Day 2

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For convenience, we will be using the Einstein summation convention for repeated indicies. This means that in any expression where letters of indicies appear more than once, a summation is applied to that index from 1 to n. With that in mind, we can write our general operator as

$$Lu := a^{ij} D_{ij} u + b^i D_i u + cu.$$

Now, we move onto the weak maximum principle for our special case operator:

$$Lu = a^{ij} D_{ij} u.$$

Theorem 0.1 (Weak Maximum Principle). *Consider $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ and $f : \Omega \to \mathbb{R}$. If $Lu \geq f$, then*

$$u \leq \max_{\partial\Omega} u + \frac{1}{2} \left(\sup_{\overline{\Omega}} \frac{|f|}{\operatorname{Tr} \mathcal{A}} \right) \cdot (\operatorname{diam} \Omega)^2 , \tag{1}$$

where $\operatorname{Tr} \mathcal{A}$ is the trace of the coefficient matrix $\mathcal{A}$ (i.e. $\operatorname{Tr} \mathcal{A} = \sum_{i=1}^{n} a^{ii}$).

Note: If $f \equiv 0$, the above reduces to the result of the first weak maximum principle we proved. Namely, $u \leq \max_{\partial\Omega} u$.

Proof: Without loss of generality we may translate Ω so that it contains the origin. Again, we consider an axillary function:

$$v = u + k|x|^2,$$

where k is an arbitrary constant to be determined later. Since the Hessian Matrix $D^2 u$ is symmetric, we may choose a coordinate system such that $D^2 u$ is diagonal. In this coordinate system, we calculate

$$Lv = Lu + 2ka^{ij} S_{ij} = Lu + 2k \cdot \operatorname{Tr} \mathcal{A},$$

where $S_{ii} = 1$ and $S_{ij} = 0$ when $i \neq j$. Now, suppose that v attains an interior maximum at $y \in \Omega$. From calculus, we then know that $D^2 v(y) \leq 0$, which implies that $a^{ij} D_{ij} \leq 0$ as $\mathcal{A}$ is positive definite. Now, if we choose

$$k > \frac{1}{2} \sup_{\overline{\Omega}} \frac{|f|}{\operatorname{Tr} \mathcal{A}},$$

our above calculation indicates that $Lv > 0$; a contradiction. Thus,

$$v \leq \max_{\partial\Omega} v$$

which implies the result as Ω contains the origin (which indicates that $|x|^2 \leq (\operatorname{diam} \Omega)^2$. ■

0.1 Application

Now we will apply the weak maximum principle to the second most famous elliptic PDE, the *Minimal Surface Equation*:

$$(1 + |Du|^2)\Delta u - D_i u D_j u D_{ij} u = 0.$$

This is a quasilinear PDE as only derivatives of the 1st order are multiplied against the 2nd order ones (a fully nonlinear PDE would have products of second order derivatives). We will now show that this is indeed an elliptic PDE. Recalling our special case $Lu = a^{ij} D_{ij} u$, we can rewrite the minimal surface equation in this form with

$$a^{ij} = (1 + |Du|^2) S_{ij} + D_i u D_j u,$$

with S being as it was in the previous proof. Now, we calculate

$$\begin{aligned} a^{ij} \xi_i \xi_j &= (1 + |Du|^2) S_{ij} \xi_i \xi_j - D_i u D_j u \xi_i \xi_j \\ &= (1 + |Du|^2)|\xi|^2 - D_i u D_j u \xi_i \xi_j \\ &= (1 + |Du|^2)|\xi|^2 - (D_i u \xi_i)^2 \\ &\geq (1 + |Du|^2)|\xi|^2 - |Du|^2 |\xi|^2 \\ &= |\xi|^2. \end{aligned}$$

Note: To get the third equality in the above, we simply relabeled repeated indicies. Since one sums over a repeated index, it's representation is a dummy index, whose relabeling does not affect the calculation.

From this calculate, we conclude this equation is indeed elliptic. Now, upon taking $f = 0$ and replacing u by $-u$, we can now apply the weak maximum principle to this equation.