Lectures on Partial Differential Equations:Day 1

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1 Course Logistics

The subject of partial differential equations (PDE) encompasses several sub-branches of mathematics. Two of the main sub-categories correspond to the study of Elliptic and Hyperbolic differential equations. The first three weeks of this course will be dedicated to the study of Elliptic equations, while the last week will be used to go over some aspects of hyperbolic equations.

- Text: *Partial Differential Equations of Second Order* by David Gilbard and Neil Trudinger.
- Outline:

Week 1: Potential Theory and Maximum Principles.

Week 2: Sobolev Spaces.

Week 3: Applications to General Equations.

2 Preliminaries

2.1 Notation

2.1.1 Set Notation

Ω will signify an open subset of Euclidean n-space, $\mathbb{R}^n$. A *domain* is understood to be a a connected, open set. In addition, we use the following standards of notation:

$$|\Omega| = \text{volume of } \Omega = \text{Lebesgue measure of } \Omega.$$
$$S \subset \mathbb{R}^n,\ \partial S = \text{ boundary of } S,\ \overline{S} = \text{ closure of } S.$$

Balls are extremely important sets in PDE theory and thus warrant their own notation. $B_R(y)$ signifies an open ball of radius R and center y. Correspondingly ω_n refers to the volume of the unit ball in $\mathbb{R}^n$. The following relation's derivation is left as an exercise,

$$\omega_n = \frac{2\pi^{n/2}}{n \cdot \Gamma\left(\frac{n}{2}\right)}.$$

Note that $\Gamma()$ is a special function which reduces to the standard factorial when its argument is a non-negative integer.

2.1.2 Notation for Derivatives

Taking $x \in \mathbb{R}^n$, i.e. $x = (x_1, \dots, x_n)$, we have the following standard notation for the partial derivative:

$$D_i = \frac{\partial u}{\partial x_i},$$

and for the gradient

$$Du = (D_1 u, \dots, D_n u)(= \nabla u).$$

Note that ∇u is classical notation as Du will often indicated a weak gradient. Correspondingly, we have for second order derivatives

$$D_{ij} u = \frac{\partial^2 u}{\partial x_i \partial x_j}.$$

Next, we define the *Hessian Matrix* of u as

$$D^2 u = [D_{ij} u],$$

where the indices i and j correspond to the rows and columns of the Hessian matrix respectively. A convenient alternative notation that sometimes lends itself to confusion is the following:

$$u_i = D_i u, \ u_{ij} = D_{ij} u.$$

The final piece of notation that we will go over is that of *multi-index notation.* Consider a vector $\beta = (\beta_1, \dots, \beta_n)$ with $0 \leq \beta_i \in \mathbb{Z}$. In the familiar way, we define the magnitude of the vector as

$$|\beta| = \sum_{i=1}^{n} \beta_i^2.$$

With that, we write

$$D^\beta u = \frac{\partial^{|\beta|} u}{\partial x_i^{\beta_1} \cdots \partial x_n^{\beta_n}}.$$

Example 1. *If $n = 3$ and $\beta = (1, 2, 0)$, we have*

$$D^\beta u = \frac{\partial^3 u}{\partial x_1 \partial x_2^2}.$$

Remark: Multi-index notation really becomes a necessity when dealing with equations with order > 2.

2.2 Classical Function Spaces

Now, we will go through the definitions of classical functions spaces used in PDE theory. First, we have spaces of continuous functions:

$$\begin{aligned} C^0(\Omega) &= \{\text{functions continuous in } \Omega\} \\ C^0(\overline{\Omega}) &= \left\{\text{functions continuous in } \overline{\Omega}\right\}. \end{aligned}$$

Next, we have spaces of functions with continuous classical derivatives:

$$\begin{aligned} k \geq 0,\ C^k(\Omega) &= \{\text{functions whose deriv. up to and including} \\ &\qquad \text{order } k \text{ are continuous in } \Omega\} \\ C^k(\overline{\Omega}) &= \{\text{functions whose deriv. of order} \leq k \text{ have} \\ &\qquad \text{continuous extensions to } \overline{\Omega}\}. \end{aligned}$$

Remark: Wording of the last definition is a bit different than saying the derivatives of order $\leq k$ continuous on $\overline{\Omega}$; this is actually a bit stronger considering certain pathological domains.

The *Support* of u is defined as

$$\operatorname{supp} u = \overline{\{x \in \mathbb{R}^n \,|\, u(x) \neq 0\}}.$$

Correspondingly, we define

$$C_0^k(\Omega) = \{\text{functions in } C^k(\Omega) \text{ having compact support in } \Omega\}.$$

In the case where $k = \infty$, we have get a very important function space from the above definition, namely

$$C_0^\infty(\Omega) = \{\text{infinitely differentiable functions with compact support in } \Omega\}.$$

Specifically, $C_0^\infty(\Omega)$ is the space of all test functions, whose importance will be demonstrated later.

3 Weak Maximum Principle

To begin this section, we introduce the *Laplacian* differential operator:

$$\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}, \;\; u \in C^2(\Omega).$$

Note that Δu is sometimes written as $\mathrm{div} \cdot \nabla u$ in classical notation. Coupled to this we have the following definition

Definition 3.1. *$u \in C^2(\Omega)$ is called* harmonic *in Ω if and only if $\Delta u = 0$ in Ω.*

Example 1. *$u(x_1, x_2) = x_1^2 - x_2^2$ is a nonlinear harmonic function in $\mathbb{R}^2$.*

In addition to the above, we have the next definition.

Definition 3.2. *$u \in C^2(\Omega)$ is called* subharmonic*(*superharmonic*) if and only if $\Delta u \geq 0 (\leq 0)$ in Ω.*

Remark: It is obvious, due to the linearity of the Laplacian, that if u is subharmonic, then $-u$ is superharmonic and vice-versa.

Example 2. *If $x \in \mathbb{R}^n$, then $\Delta |x|^2 = 2n$. Thus, $|x|^2$ is subharmonic in $\mathbb{R}^n$.*

Now, we come to our first theorem

Theorem 3.3 (Weak Maximum Principle for the Laplacian). *Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ with Ω bounded in $\mathbb{R}^n$. If $\Delta u \geq 0$ in Ω, then*

$$u \leq \max_{\partial\Omega} u \;\; \left(\Longleftrightarrow \;\; \max_{\overline{\Omega}} u = \max_{\partial\Omega} u \right). \tag{1}$$

Conversely, if $\Delta u \leq 0$ in Ω, then

$$u \geq \min_{\partial\Omega} u \;\; \left(\Longleftrightarrow \;\; \min_{\overline{\Omega}} u = \min_{\partial\Omega} u \right). \tag{2}$$

Consequently, if $\Delta u = 0$ in Ω, then

$$\min_{\partial\Omega} u \leq u \leq \max_{\partial\Omega} u. \tag{3}$$

Proof: We will just prove the subharmonic result, as the superharmonic case is proved the same was. Given u subharmonic, take $\epsilon > 0$ and define $v = u + \epsilon|x|^2$, so that $\Delta v = \Delta u + 2\epsilon n > 0$ in Ω. If v takes attains a maximum in Ω, we have $\Delta v \leq 0$ at that point, a contradiction. Thus, by the compactness of Ω, we have

$$v \leq \max_{\partial\Omega} v.$$

Finally, we take $\epsilon \to 0$ in the above to get the result. ■

Corollary 3.4 (Uniqueness). *Take $u, v \in C^2(\Omega) \cap C^0(\overline{\Omega})$. If $\Delta u = \Delta v = 0$ in Ω and $u = v$ on $\partial\Omega$, then $u = v$ in Ω.*

Before moving onto more general elliptic equations, we briefly remind ourselves of the *Dirichlet Problem*, which takes the following form.

$$\begin{aligned} &\Delta u = 0 \text{ in } \Omega \\ &u = \phi \text{ on } \partial\Omega \text{ for some } \phi \in C^0(\partial\Omega) \end{aligned}$$

4 Linear Elliptic Operators

Again we take $\Omega \subset \mathbb{R}^n$. We now consider the following differential operator:

$$Lu := \sum_{i,j=1}^{n} a^{ij} D_{ij} u + \sum_{i=1}^{n} b^i D_i u + cu,$$

where $a^{ij}, b^i, c : \Omega \to \mathbb{R}$. Now, we state the following.

Definition 4.1. *The operator L is said to be* elliptic(degenerate elliptic) *if $\mathcal{A} := [a^{ij}] > 0 (\geq 0)$ in Ω.*

Remarks:

i.) In the above definition $\mathcal{A} > 0$ means the minimum eigenvalue of the matrix $\mathcal{A}$ is > 0. More explicitly, this means that

$$\sum_{i,j=1}^{n} a^{ij} \xi_i \xi_j > 0, \ \forall \xi \in \mathbb{R}^n.$$

ii.) The last definition indicates that parabolic PDE (such as the Heat Equation) are really degenerate elliptic equations.