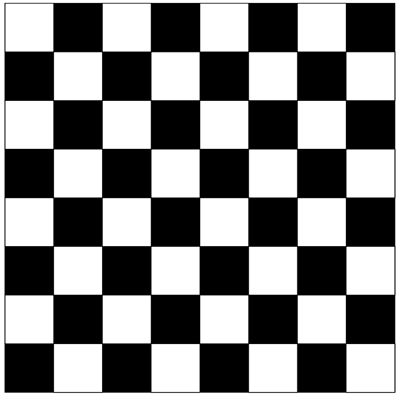
Examples of combinatorial matrices:



## Latin squares

| | | | | | | |
|---|---|---|---|---|---|---|
| **7** | **1** | **6** | **3** | **2** | **5** | **4** |
| **1** | **2** | **3** | **4** | **5** | **6** | **7** |
| **5** | **7** | **2** | **1** | **3** | **4** | **6** |
| **4** | **6** | **7** | **2** | **1** | **3** | **5** |
| **2** | **3** | **4** | **5** | **6** | **7** | **1** |
| **6** | **5** | **1** | **7** | **4** | **2** | **3** |
| **3** | **4** | **5** | **6** | **7** | **1** | **2** |

Each symbol occurs exactly once in each row and exactly once in each column.

## Latin rectangles

| | | | | | | |
|---|---|---|---|---|---|---|
| **7** | **1** | **6** | **3** | **2** | **5** | **4** |
| **1** | **2** | **3** | **4** | **5** | **6** | **7** |
| **5** | **7** | **2** | **1** | **3** | **4** | **6** |

Each symbol occurs exactly once in each row and *at most* once in each column.

## Hadamard matrices

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| + | + | + | + | + | + | + | + |
| + | − | + | + | − | + | − | − |
| + | − | − | + | + | − | + | − |
| + | − | − | − | + | + | − | + |
| + | + | − | − | − | + | + | − |
| + | − | + | − | − | − | + | + |
| + | + | − | + | − | − | − | + |
| + | + | + | − | + | − | − | − |

Between any two rows the number of places where they match equals the number of places where they differ.

A Hadamard matrix $H$ of order $n$ satisfies $HH^T = nI$, where $I$ is the identity matrix.

## Orthogonal matrices

Between any pair of columns each possible ordered pair of symbols occurs the same number $\lambda$ of times. Throughout this course we will assume $\lambda = 1$.

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & 2 & 2 \\ 1 & 0 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 2 & 2 \\ 2 & 0 & 1 \\ 2 & 0 & 2 \\ 2 & 1 & 0 \\ 2 & 1 & 2 \\ 2 & 2 & 0 \\ 2 & 2 & 1 \end{pmatrix}$$

## Binary matrices (also called $(0,1)$-matrices)

$$\begin{matrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \end{matrix}$$

Every entry is either 0 or 1.

The set $\Lambda_n^k$ denotes the $(0,1)$-matrices of order $n$ with exactly $k$ ones in each row and column.

**Example:** $\Lambda_8^3$ contains:

$$\begin{matrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \end{matrix}$$

By $J_n$ we denote the unique matrix in $\Lambda_n^n$ (every entry is 1).

For any $A \in \Lambda_n^k$ we define $\overline{A} = J_n - A \in \Lambda_n^{n-k}$ to be the complementary matrix.

$$\begin{matrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \end{matrix} \qquad \begin{matrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \end{matrix}$$

$A \in \Lambda_8^3$ $\qquad$ $\overline{A} \in \Lambda_8^5$

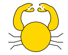
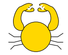
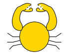
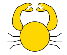
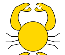
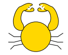
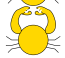
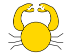
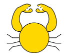
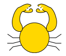
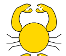
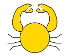
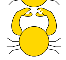
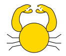
## Permutation matrices

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |

$\Lambda_n^1$ – exactly one 1 in each row and column.

The $1$ entries of a permutation matrix form a **diagonal**.
In a general $n \times n$ matrix, we say that a diagonal is a set of $n$ entries no two of which share a row or a column.

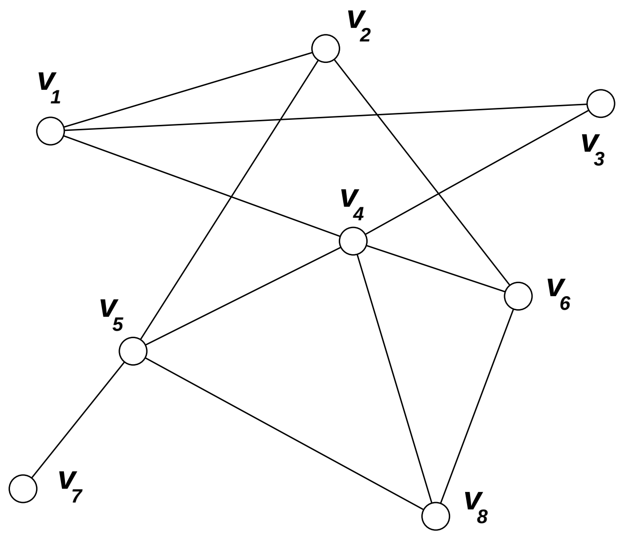
## Frequency squares

| | | | | | | |
|---|---|---|---|---|---|---|
| $f$ | $f$ | $f$ | 5 | 4 |  |  |
| $f$ | 4 | 5 |  |  | $f$ | $f$ |
|  | $f$ | 4 | $f$ | 5 |  | $f$ |
|  | $f$ | $f$ | 4 | $f$ | 5 |  |
| 4 | 5 |  |  | $f$ | $f$ | $f$ |
| $f$ |  | $f$ | $f$ |  | 4 | 5 |
| 5 |  |  | $f$ | $f$ | $f$ | 4 |

Each row and column contains the same symbols, with the same multiplicities.

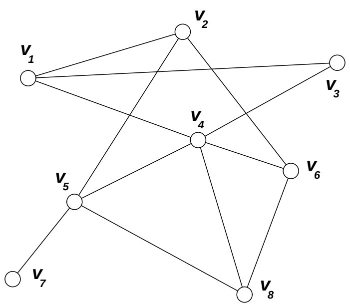
Includes latin squares and $\Lambda_n^k$ as special cases.

## A graph




Formally, a graph is a set $V$ of **vertices** and a set $E$ of **edges**, where each edge is an unordered pair of vertices.
The two vertices in an edge are said to be **adjacent**.
Two edges are **adjacent** if they share a vertex.
A graph $G$ is completely described by its **adjacency matrix**, which is a $(0, 1)$-matrix with rows and columns indexed by the vertices of $G$ and with the entry in row $u$, column $v$ being $1$ if $u$ and $v$ are adjacent in $G$ and $0$ otherwise.

| | $v_1$ | $v_2$ | $v_3$ | $v_4$ | $v_5$ | $v_6$ | $v_7$ | $v_8$ |
|---|---|---|---|---|---|---|---|---|
| $v_1$ | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 0 |
| $v_2$ | 1 | 0 | 0 | 0 | 1 | 1 | 0 | 0 |
| $v_3$ | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| $v_4$ | 1 | 0 | 1 | 0 | 1 | 1 | 0 | 1 |
| $v_5$ | 0 | 1 | 0 | 1 | 0 | 0 | 1 | 1 |
| $v_6$ | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 1 |
| $v_7$ | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| $v_8$ | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 |



For a given vertex $v$ the number of other vertices adjacent to $v$ is called the **degree** of $v$. If every vertex of a graph has the same degree, say $k$, then we say the graph is **$k$-regular** (or just **regular**).

By definition, the adjacency matrix of a graph is symmetric and has zeroes on the diagonal. If the graph is $k$-regular then each row and column of the adjacency matrix has total $k$.

Another way to record the structure of a graph $G$ by using a $(0, 1)$-matrix, is to use the **incidence matrix**. This matrix has rows corresponding to the vertices of $G$ and columns corresponding to the edges of $G$. There are exactly two $1$'s in the column corresponding to each edge, indicating which two vertices form the edge.

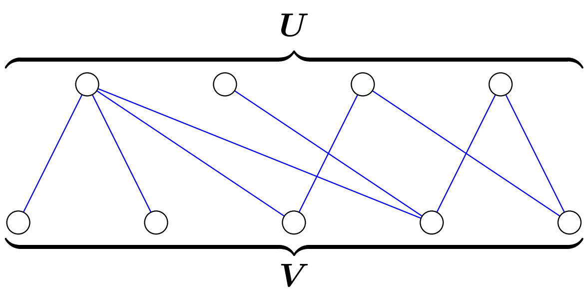
## Bipartite graphs

A graph $G$ is **bipartite** if its vertices can be partitioned into two disjoint subsets $U$ and $V$ such that every edge in $G$ joins a vertex in $U$ to a vertex in $V$.

In this case we call $(U, V)$ a **bipartition**.

A graph with bipartition $(U, V)$ corresponds to a $(0, 1)$-matrix (called the **biadjacency matrix**) in which the rows are indexed by the vertices in $U$ and the columns are indexed by the vertices in $V$. Again the entry in row $u$, column $v$ is $1$ if $u$ and $v$ are adjacent in $G$ and $0$ otherwise.

**Example:** A bipartite graph and its corresponding biadjacency matrix.




$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

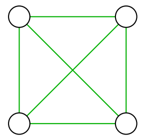
If a bipartite graph is $k$-regular then its biadjacency matrix belongs to $\Lambda_n^k$, where $n$ is the number of vertices in each part of the bipartition.

A **complete graph** is a graph in which every pair of vertices is adjacent.
If it has $n$ vertices it is traditionally denoted $K_n$.

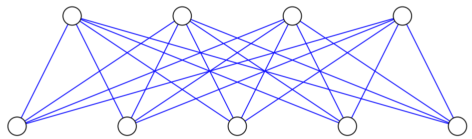
A **complete bipartite graph** with bipartition $(U, V)$ is a graph in which every vertex in $U$ is adjacent to every vertex in $V$ (and there are no other edges).

If $|U| = r$ and $|V| = s$ the graph is denoted $K_{r,s}$.

Hence $K_{n,n}$ has biadjacency matrix $J_n$.



$K_4$



$K_{4,5}$