DIFFERENT SIZES OF INFINITY

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Moral: Extending a new idea to its logical conclusion can lead to surprising and counterintuitive outcomes as well as a more accurate view of reality.

The Dodge Ball Game.

Diagonalising Out of a Sequence

- Friend writes list of 7 natural numbers, each 7 digits long.

$\boxed{5}$	6	9	9	7	8	5
3	$\boxed{4}$	0	7	4	6	7
4	6	$\boxed{7}$	9	0	6	8
4	2	3	$\boxed{5}$	6	7	5
3	9	1	5	$\boxed{6}$	0	9
1	3	7	7	4	$\boxed{6}$	5
8	7	6	9	6	8	$\boxed{9}$

- easy to find another 7 digit number n not in list.
- BUT — systematic approach, a là Dodge Ball

Method

(1) first digit in n selected to be different from first digit in diagonal.
(2) second digit in n selected to be different from second digit in the diagonal.
(3) third digit in n selected to be different from third digit in the diagonal.
(4) Etc.

Might end with $n = 4324012$, for example.

Note:

(1) After first step we know n will be different from first number in list. Does not matter how we select later digits in n.
(2) After 2nd step we know n will be different from second number in list, no matter how we select other digits in n.
(3) After 3rd step we know n will be different from third number in the list, no matter how we select other digits in n.
(4) Etc.

$\mathbb{R}$ is not Countably Infinite.

- What does this mean?
- Following does *not* list all real numbers.

$$0, .1, .2, \ldots, 1, -.1, -.2, \ldots, -1, 1.1, 1.2, \ldots, 2, -1.1, -1.2, \ldots, -2, \ldots.$$

  What does this prove? (Not much!)
- How *can* we prove $\mathbb{R}$ is not Countably Infinite?
- *Answer*: Must show that for *every* infinite sequence of real numbers, some real number is missing
- *Method*: Extend the Dodge Ball Game!

Theorem. $\mathbb{R}$ *is not countably infinite.*

Proof Sketch.

- Suppose $s_1, s_2, \ldots, s_n, \ldots$ is a sequence of real numbers.
- We will show

there is another real number r not in the sequence. ($\bigstar$)

Once we have done this, the Theorem is proved. *Why?*

- Write

$$\begin{aligned} s_1 &= a_1.\boxed{a_{11}}a_{12}a_{13}a_{14}\ldots a_{1n}\ldots \\ s_2 &= a_2.a_{21}\boxed{a_{22}}a_{23}a_{24}\ldots a_{2n}\ldots \\ s_3 &= a_3.a_{31}a_{32}\boxed{a_{33}}a_{34}\ldots a_{3n}\ldots \\ s_4 &= a_4.a_{41}a_{42}a_{43}\boxed{a_{44}}\ldots a_{3n}\ldots \\ &\vdots \\ s_n &= a_n.a_{n1}\,a_{n2}\,a_{n3}a_{n4}\ldots\boxed{a_{nn}}\ldots \\ &\vdots \end{aligned} \tag{1}$$

Example: $s_1 = 17.325168432\ldots$ and $s_2 = -0.298461705\ldots$ then

$$a_1 = 17, a_{11} = 3, a_{12} = 2, a_{13} = 5, a_{14} = 1, a_{15} = 6, \ldots,$$
$$a_2 = -0, a_{21} = 2, a_{22} = 9, a_{23} = 8, a_{24} = 4, a_{25} = 6, \ldots .$$

- Use Dodge Ball with 4 and 6 (any choices OK, but don't use 9 or 0) to get

$$r = .r_1r_2r_3r_4\ldots r_n\ldots$$

not in sequence, as follows:

(1) If $a_{11} \neq 4$ then $r_1 = 4$, if $a_{11} = 4$ then $r_1 = 6$.
(2) If $a_{22} \neq 4$ then $r_2 = 4$, if $a_{22} = 4$ then $r_2 = 6$.
(3) If $a_{33} \neq 4$ then $r_3 = 4$, if $a_{33} = 4$ then $r_3 = 6$.
(4) If $a_{44} \neq 4$ then $r_4 = 4$, if $a_{44} = 4$ then $r_4 = 6$.
$\vdots$
(n) If $a_{nn} \neq 4$ then $r_n = 4$ and if $a_n = 4$ then $r_n = 6$.
$\vdots$.

Then r is different from every real number in the sequence!!

- This proves ($\bigstar$) and so completes the proof. □

Comments and Objections?

⋆Uncountable Sets.

Definition.

If A can be put in one-to-one correspondence with $\mathbb{N}$, it has *cardinality* d.
If A can be put in one-to-one correspondence with $\mathbb{R}$, it has *cardinality* c.
d and c are called *cardinals* or *cardinal numbers*.

A is *countable* if it is finite or has cardinality d.

A is *uncountable* if it is infinite and does not have cardinality d.

Common Error: Uncountable sets *cannot* be written

$$A = \{a_1, a_2, a_3, \ldots, a_n, \ldots\}.$$

⋆Removing Part of an Infinite Set.

General Result: If from an infinite set B a subset A of smaller cardinality is removed, then the remaining set has the same cardinality as B.

We prove the following cases:

Theorem.

(1) *Suppose B is an infinite set and A is a finite subset. Then the set $B \setminus A$ has the same cardinality as B.*
(2) *Suppose B is an uncountably infinite set and A is a countably infinite subset. Then the set $B \setminus A$ has the same cardinality as B.*

Proof. *** Discussion *** □

⋆The Set $\mathbb{I}$ of Irrational Numbers.

Theorem. *The set $\mathbb{I}$ of irrational numbers is uncountable.*

Proof.

$$\mathbb{I} = \mathbb{R} \setminus \mathbb{Q}.$$

Since $\mathbb{R}$ is uncountable and $\mathbb{Q}$ is countable, it follows from previous Theorem that $\mathbb{I}$ and $\mathbb{R}$ have the same cardinality, namely c. □