A rational paradox - Chris Wetherell

Numbers

We will assume there are such things, e.g.

- Natural numbers $\mathbf{N} = \{1, 2, 3, 4, 5, \dots\}$
- Integers $\mathbf{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$
- Rational numbers $\mathbf{Q} = \{a/b \mid a \in \mathbf{Z}, b \in \mathbf{N}\}$
- Real numbers $\mathbf{R} =$ all of the above and everything else

How many rationals are there?

Theorem. *There are lots and lots and lots of rational numbers, but there really aren't very many at all.*

Proof. This is an immediate consequence of Lemmas 1 and 2. □

Density of rationals

Lemma 1. *There are lots and lots and lots of rational numbers because they are dense in the reals numbers. That is, between any two reals there are infinitely many rationals.*

Proof. It's enough to find just one rational in between. Compare decimal expansions. □

Cardinality

Definition. Sets A and B have the same *size* or *cardinality* if their elements can be paired up via a one-to-one correspondence.

Notation. We write $|A| = |B|$.

Examples. $|\mathbf{N} \setminus \{1\}| = |\mathbf{N}|$, $|2\mathbf{N}| = |\mathbf{N}|$, $|\mathbf{Z}| = |\mathbf{N}|$.

Countability

Definition. A set is *countable* if

- it is empty; or
- it has the same cardinality as $\{1, \ldots, n\}$; or
- it has the same cardinality as $\mathbf{N}$.

Notation. Write $|A| = 0$, $|A| = n$ or $|A| = \aleph_0$ respectively, and call 0, n and $\aleph_0$ *cardinals*.

Countability of rationals

Lemma 2. *There really aren't very many rationals at all because they are countable.*

Proof. To prove countability for any set, it's enough to put its elements in a list (either finite or infinite). By Lemma 1 (density), we *cannot* try to use the 'natural' order on $\mathbf{Q}$, but there are many other possible lists. □

The paradox

Corollary. *Between every pair of integers there are infinitely many rationals, but there are actually the same 'number' of rationals as integers.*

Is this really a paradox? No, because the concepts of 'density' and 'cardinality' are used to describe very different qualities.

Properties of countable sets

Every subset of a countable set is countable.

The union of two countable sets is countable.

So is the union of three countable sets.

And four. And five. And so on…

In fact, the union of *countably many* countable sets is still countable!