

**THE AUSTRALIAN NATIONAL UNIVERSITY**

**MATH2306: Mathematical Methods I  
Advanced Vector Calculus**

**Lecture Notes 2006**

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**MATH2306: MATHEMATICAL METHODS I**  
– SYLLABUS –

**Partial Differential Equations**

**Lecture 1**

- What is a PDE?
- First order PDE's in 2 variables

**Lecture 2**

- Second order PDEs: heuristic discussion

**Lecture 3**

- Second order PDEs (continued): heuristic discussion, Initial and boundary problems, well-posed problems, types of 2nd order PDE's

**Lecture 4**

- Wave equation: derivation of the wave equation for a piece of string

**Lecture 5**

- Wave equation (continued): causality, conservation of energy
- Diffusion equation: maximum principle, uniqueness of solution

**Lecture 6**

- Diffusion equation (continued): energy method, stability, diffusion on the whole line (Green's function)

**Lecture 7**

- Reflections and sources: diffusion on the half-line, reflections of waves, diffusion with a source, operator approach

**Lecture 8**

- Reflections and sources (continued): wave equation with a source, characteristic coordinates, Stokes' theorem, operator method
- Boundary value problems: wave eqn separation of variables (Dirichlet boundary conditions)

**Lecture 9**

- Boundary value problems (continued): wave eqn separation of variables (Neumann boundary conditions), diffusion equation, mixed boundary conditions
- Fourier series: sine series, cosine series, full Fourier series

**Lecture 10**

- Fourier series (continued): complex form, orthogonality and general Fourier series, complex eigenvalues, completeness, notions of convergence

**Lecture 11**

- Fourier series (continued):  $L^2$ -convergence, pointwise convergence

**Lecture 12**

- Fourier series (continued): Uniform convergence

**Lecture 13**

- Harmonic functions
- Laplace equation: maximum principle, uniqueness of the Dirichlet problem, invariance, 2D Laplacian in polar coordinates

**Lecture 14**

- Laplace equation (continued): 3D Laplace equation in polar coordinates, examples

**Lecture 15**

- Laplace equation (continued): Poisson formula, mean value theorem, strong maximum principle, differentiability of solution, Laplace equation on exterior of a disk

**Lecture 16**

- Laplace equation (continued): Laplace equation on annulus, Laplace equation on wedge
- Green's identities and Green's functions: Green's first identity, volume of ball/sphere in  $n$  dimensions

**Lecture 17**

- Green's identities and Green's functions (continued): Mean value theorem, strong maximum principle, uniqueness of Dirichlet problem, Dirichlet principle, Green's second identity, representation formula

**Lecture 18**

- Green's identities and Green's functions (continued): Green's function, solution to the Poisson problem, Dirichlet problem on a half-space

**Lecture 19**

- Green's identities and Green's functions (continued): Dirichlet problem on a ball
- Wave equation in space-time: energy functional, causality

**Lecture 20**

- Wave equation in space-time (continued): wave equation in 3D

**Lecture 21**

- Wave equation in space-time (continued): wave equation in 2D
- Diffusion equation in  $n$  dimensions
- Eigenvalue equation for the Laplacian, minimum principle

**Lecture 22**

- Distributions: definition, convergence of distributions, derivative, generalization to  $n$  dimensions
- Green's functions revisited: Laplace equation

**Lecture 23**

- Fourier transform: definition, properties, generalization to  $n$  dimensions

**Lecture 24**

- Green's functions revisited: Diffusion equation, wave equation, Laplace equation