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Monge-Ampère and related equations and their applications.

Monge-Ampère equations are second order partial differential equations of the general form $\det D^2u = f(x, u, Du)$, although more general forms arise in certain applications. Here u is a real valued function defined on some open set in Euclidean space and D^2u is the matrix of second derivatives of u . The key feature is the presence of the determinant of the second derivatives, or sometimes of some more complicated matrix involving u and its first and second derivatives. Since $\det D^2u$ is the Jacobian of the gradient map Du, these equations often have some connection with mapping problems, and with problems involving the Gauss curvature of a surface. The following three topics give should give some idea of the kinds of problems that lead to Monge-Ampère equations.

(i) Surfaces of prescribed Gauss curvature. The basic problem is to understand under what conditions it is possible to find a convex surface of prescribed Gauss curvature which also satisfies some boundary conditions. It is also interesting to look at these questions for curvatures other than the mean and Gauss curvatures, but this is more difficult and less well understood.

(ii) Isometric embedding of two dimensional Riemannian manifolds. A two dimensional Riemannian manifold is an abstract surface sitting nowhere in particular, but which somehow has the structures imposed on it that a surface gets by sitting in Euclidean space, such as tangent spaces, a metric etc. The question is whether such an abstract surface can be realized as a surface in three dimensional Euclidean space. This problem can be reduced to solving a certain Monge-Ampère equation.

(iii) Mass transport problems. The basic question is whether a given unit mass distribution can be mapped in an optimal way to another given unit mass distribution. More simply, what is the easiest way of shovelling a pile of soil into a hole in the ground? This is a question that was first formulated by Monge over two hundred years ago, and which has recently received a lot of attention. It has connections with many areas of mathematics, particularly partial differential equations, measure theory and convexity theory.

An introduction to Monge-Ampère equations (and more generally to fully nonlinear elliptic equations) is Chapter 17 of Elliptic Partial Differential Equations of Second Order, by D. Gilbarg and N.S. Trudinger (you don't need to be completely familiar with the first 16 chapters, but some background in partial differential equations is certainly necessary). This book doesn't contain much geometry, but it is good for some of the analytic aspects. The Minkowski Multidimensional Problem, by A.V. Pogorelov is also a good introduction to Monge-Ampère equations, and to some related geometric problems.

The isometric embedding problem is a large topic with many interesting connections to partial differential equations. Only a small part of this is connected to Monge-Ampère equations, and there is no really good exposition of this. A good place to start is L. Nirenberg's paper on the Weyl and Minkowski problems [Comm. Pure Appl. Math. 6. (1953), 337–394] (just the introduction to see what it is about). Pogorelov's book Extrinsic Geometry of Convex Surfaces is also worth looking at.

I don't know of a really good introduction to mass transfer problems. There is a book by S.T. Rachev called Probability Metrics and the Stability of Stochastic Models, but it is difficult to read and is mostly probability theory. There is also a recent book by Rachev called Mass Transportation Problems (in two volumes). These books give some discussion of the large range of applications. A very well written paper by W. Gangbo and R. McCann, The Geometry of Optimal Transportation [Acta Math. 177 (1996), 113161] is worth looking at to get a feeling for what it is all about. There are also several sets of lecture notes by various people dealing with different aspects of the theory.