SUGGESTED HONOURS PROJECTS FOR 2009

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1. The Yau and Lawson conjectures for minimal tori

This project concerns an important conjecture which appears to have been solved recently: It concerns minimal surfaces, in particular minimal submanifolds of spheres. It combines PDE and geometry, though the PDE required is not very much. It does involve some basic spectral theory for the Laplacian on a Riemannian manifold.

There are many examples known of submanifolds in spheres which are minimal, in the sense that their mean curvature vanishes (equivalently, they are critical points for the area functional). The simplest are of course the totally geodesic submanifolds (great spheres). The next simplest is the Clifford torus, which is the product $S^1 \times S^1$ in the three sphere $S^3 \subset R^2 \times R^2$. It is known that there are embedded minimal surfaces of every genus in S^3 (first proved by Blaine Lawson in 1970), and there are infinitely many immersed minimal tori in S^3.

The spectral theory comes in with the following observation: A submanifold (of dimension n) of a sphere is minimal if and only if each component of the position vector is an eigenfunction of the Laplacian with eigenvalue n.

Lawson conjectured in his 1970 paper that the only embedded minimal torus in S^3 is the Clifford torus (there is a related conjecture for higher genus minimal embedded surfaces). Much later Yau conjectured that an embedded minimal submanifold of dimension n has first eigenvalue equal to n (that is, there is no eigenvalue less than n). The latter is false if the submanifold is immersed rather than embedded.

The project would aim to discuss:

- the construction of minimal surfaces by Lawson;
- partial results towards the Yau conjecture by Choi and Wang in 1983 (the first eigenvalue is at least $n/2$) — it is tempting to think that further progress could be made here by modifying their methods;
- the result by Montiel and Ros from 1985 that the only minimal torus in S^3 with first eigenvalue equal to 2 is the Clifford torus (the latter implies that the Lawson conjecture is equivalent to the Yau conjecture for tori);
- the recent (2007) proof by Pimentel of the Lawson conjecture (this is not yet published so the idea would be to read the preprint carefully and try to decide whether it is correct).

References:

(1) F. Pimentel, *A proof of the Lawson conjecture for minimal tori embedded in S^3*, `arXiv:math/0703136v2 [math.DG]`
(2) H. B. Lawson, Jr., *Complete minimal surfaces in S^3*, Annals of Mathematics **92** (1970), 335-374 (available online)

(3) S. Montiel and A. Ros, *Minimal immersions of surfaces by the first eigenfunctions and conformal area*, Inventiones Mathematicae **83** (1986), 153-166 (available online)
(4) H. I. Choi and A. N. Wang, *A first eigenvalue estimate for minimal hypersurfaces*, Journal of Differential Geometry **18** (1983), 559-562.

2. Ricci flow: Four-manifolds with positive isotropic curvature

The Ricci flow is a geometrically defined heat equation which deforms Riemannian metrics. It was first introduced by Richard Hamilton in 1982, in a landmark paper which proved that any compact simply connected three-dimensional manifold with positive Ricci curvature is diffeomorphic to the standard sphere. Since then it has been used to prove many important results, including most spectacularly the proof of the Poincaré conjecture and the Geometrization conjecture by Perel'man, and the recent classification of manifolds with positive curvature operator by Böhm and Wilking, and the proof of the differentiable 1/4-pinching sphere theorem by Brendle and Schoen.
There is a natural conjecture not yet solved which seems suited for application of the Ricci flow: Micallef and Moore introduced a new 'positive curvature' condition, called 'positive curvature on totally isotropic 2-planes', or 'positive isotropic curvature' (PIC) for short. They proved that any simply connected manifold with PIC is *homeomorphic* to a sphere. The question is, can this be improved to give *diffeomorphic*? Equivalently, can there exist an exotic sphere (a manifold homeomorphic but not diffeomorphic to a sphere) with PIC?
This is so far unsolved, though recent progress by Brendle and Schoen and Nguyen shows that at least Ricci flow preserves the PIC condition. The one case where the result is known is the four-dimensional case, where the algebra of curvature is relatively tractable (the condition is vacuous in two and three dimensions). The idea of the project would be to present the required background concerning Ricci flow, and work through the proof in this case, using the original paper by Hamilton and two more recent papers (which also use some of the methods of Perel'man). In particular the argument uses Ricci flow with surgery, an idea which Perel'man adopted in his proof of the Poincaré conjecture. There is scope for publishable new work on this project in several directions.
References:

(1) R. S. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Differential Geometry **17** (1982), 255–306.
(2) R. S. Hamilton, *Four-manifolds with positive curvature operator*, J. Differential Geometry **24** (1986), 153–179.
(3) M. Micallef and J. D. Moore, *Minimal two-spheres and the topology of manifolds with positive curvature on totally isotropic two-planes*, Ann. Math. **127** (1988), 199–227.
(4) R. S. Hamilton, *Four-manifolds with positive isotropic curvature*, Comm. Anal. Geom. **5** (1997), 1–92.
(5) B.-L. Chen and X.-P. Zhu, *Ricci flow with surgery on four-manifolds with positive isotropic curvature*, J. Differential Geometry **74** (2006), 177–264. `arXiv:0810.1999v1 [math.DG]`

(6) B.-L. Chen, S.-H. Tang and X.-P. Zhu, *Complete classification of compact four-manifolds with positive isotropic curvature*, `arXiv:0810.1999v1 [math.DG]`

3. Deforming diffeomorphisms of the sphere to rotations

It is a theorem of Smale that the group $\mathrm{Diff}_+(S^2)$ of orientation-preserving diffeomorphisms of the two-dimensional sphere has the rotation group $SO(3)$ as a strong deformation retract. That is, there exists a continuous map $f : \mathrm{Diff}_+(S^2) \times [0,1] \to \mathrm{Diff}_+(S^2)$ with $f(\varphi, 0) = \varphi$ and $f(\varphi, 1) \in SO(3)$ for any $\varphi \in \mathrm{Diff}_+(S^2)$, and $f(\varphi, t) = \varphi$ for all t if $\varphi \in SO(3)$. This says In particular that any orientation-preserving diffeomorphism of S^2 can be continuously deformed through diffeomorphisms to a rotation.
In this project the idea is to look at a proof of this result which uses a geometric heat equation to deform the initial diffeomorphism to a rotation. The proof uses the mean curvature flow to deform the graph of the diffeomorphism (a two-dimensional submanifold in $S^2 \times S^2$) to the graph of a rotation. It gives something extra, namely an equivariant strong deformation retraction, but works only on the space of area-preserving oriented diffeomorphisms.
The project would present the required background for mean curvature flow, and the proof by Mu-Tao Wang. Possible further directions involving new work could include investigating the use of other flows to remove the area-preserving assumption, or to prove the related result that the space of oriented diffeomorphisms has an equivariant strong deformation retract to the space of Möbius transformations. It could also present related results on the mean curvature flow of submanifolds.
References:

(1) S. Smale, *Diffeomorphisms of the 2-sphere*, Proc. Amer. Math. Soc. **10** (1959), 621-626.
(2) Mu-Tao Wang, *Deforming area-preserving diffeomorphisms of surfaces by mean curvature flow*, Math. Res. Lett. **8** (2001), no.5-6, 651-662. `arXiv:math/0110020v1 [math.DG]`

4. Isospectral metrics

Back in 1966 Mark Kac asked the question “Can one hear the shape of a drum?” — that is, if you know all of the eigenvalues of the Laplacian of a domain, can you determine what the domain is? The question applies for domains in the plane, or for manifolds.
The question for manifolds was answered by John Milnor, who found two 16-dimensional non-isometric Riemannian manifolds with identical spectra. Kac’s question for planar domains was answered in 1992 by Carolyn Gordon, David Webb and Scott Wolpert, who found examples of planar domains with the same spectrum.
The idea of this project would be to work through this construction, which actually uses results about actions of Lie groups on manifolds, even in the planar case. There are also many more recent related results which you could lead into in this project — generally speaking, one can ask which properties of a manifold or domain one can ‘hear’, i.e. determine from the spectrum, and there are many ways of doing this.
References:

(1) Kac, M. *Can One Hear the Shape of a Drum?* Amer. Math. Monthly **73**, 1-23, 1966.
(2) Gordon, C.; Webb, D.; and Wolpert, S. *Isospectral Plane Domains and Surfaces via Riemannian Orbifolds*, Invent. Math. **110**, 1-22, 1992.
(3) C. Gordon, D. Webb and S. Wolpert, *You Cannot Hear the Shape of a Drum.*, Bull. Amer. Math. Soc. **27**, 134-138, 1992. `http://arxiv.org/pdf/math/9207215`

5. Inequalities for fundamental frequencies

There are many interesting inequalities concerning the spectrum of the Laplacian (on domains or on manifolds, with Dirichlet or Neumann boundary conditions). A very interesting one is the Payne-Polya-Weinberger conjecture, which states that for a smooth bounded domain in $\mathbb{R}^n$, the Laplacian with Dirichlet boundary has λ_2/λ_1 no greater than the value for the ball. This was proved by Ashbaugh and Benguria.
The project would present the required spectral theory and analysis to understand the problem, and present the proof. It could also present a variety of other related results on the eigenvalues of the Laplacian, such as the fundamental gap conjecture.
References:

(1) Mark Ashbaugh and Rafael Benguria, *Proof of the Payne-Pólya-Weinberger Conjecture*, Bull. Amer. Math. Soc. **25** (1991), 19–29.
(2) Mark Ashbaugh and Rafael Benguria, *A sharp bound for the ratio of the first two eigenvalues of Dirichlet Laplacians and extensions*, Ann. of Math. (2) **135** (1992), no. 3, 601–628.
(3) Mark Ashbaugh and Rafael Benguria, *A second proof of the Payne-Plya-Weinberger conjecture*, Comm. Math. Phys. **147** (1992), no. 1, 181–190.

The above projects are suggestions only, and there are many more possibilities! I am happy to discuss other ideas for projects involving differential geometry and/or partial differential equations and the calculus of variations.