POSSIBLE HONOURS PROJECTS

ANDREW HASSELL

1. **Eigenfunction concentration.** Suppose a domain Ω in $\mathbb{R}^2$ with smooth boundary is given. A *Dirichlet* or *Neumann* eigenfunction on Ω is a function on $\overline{\Omega}$, the closure of Ω, that satisfies the equation $-\Delta u = \lambda u$ inside the domain and either the Dirichlet boundary condition $u = 0$, or the Neumann boundary condition, $\partial_\nu u = 0$, at the boundary. Here $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplacian, and ν is the unit normal vector at the boundary. It turns out that there is a discrete set of eigenfunctions λ, which can be arranged in a sequence $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \dots \to \infty$ for which there is a nontrivial solution to the eigenfunction equation, and corresponding eigenfunctions u_1, u_2, $\dots$. These eigenfunctions have some pleasant properties; for example, they can be chosen to form on orthonormal basis of the Hilbert space $L^2(\Omega)$.

It turns out that asymptotic properties of the eigenfuctions u_n, as $n \to \infty$, is intimately related to the dynamical properties of the billiard flow. The billiard flow is a dynamical system whose state space is the set of unit tangent vectors in Ω. The flow at time t moves a unit tangent vector along the billiard trajectory starting at that tangent vector (straight line motion, bouncing off the boundary obeying the usual law of reflection) for distance t. For example, if the flow is chaotic then the eigenfunctions become equidistributed in the sense that the probability measures u_n^2 tend weakly to uniform measure on Ω. There are many questions to consider, such as

- How big can eigenfunctions u_n be in L^p, given that they are normalized so that $\|u_n\|_{L^2} = 1$ (as a function of the eigenvalue)? How big can they be when restricted to submanifolds?
- Can eigenfunctions concentrate (asymptotically, as $n \to \infty$) in subregions of the domain?
- If the billiard flow has a certain dynamical property (such as ergodicity, chaos, integrability), how is this visible in the behaviour of eigenfunctions?
- Inverse results: what geometric or dynamical properties of the domain can be deduced from the eigenfunctions or eigenvalues?

Representative paper: A. Hassell and T. Tao, Upper and lower bounds for normal derivatives of Dirichlet eigenfunctions. Math. Res. Lett. 9 (2002), 289-307.

2. **Wave diffraction.** Wave diffraction is the phenomenon that solutions $u(x, t)$ of the wave equation

$$\partial_t^2 u = \sum_{i=1}^{n} \partial_{x_i}^2 u$$

when they meet an corner of their domain (or some other form of conic singularity). A singularity of the wave equation that hits the corner will produce an outgoing circular wave emanating from the corner. This phenomenon has been studied for well over 100 years, but surprisingly, there are still things remaining to be proved about diffraction on a domain as simple as a planar wedge.

The idea of this project would be to try to relate the outgoing wave emanating from the corner to properties of the incoming wave and and the 'link' (i.e. circular cross-section) of the corner.

A classic paper in this area is Keller, Joseph B.; Blank, Albert Diffraction and reflection of pulses by wedges and corners. Comm. Pure Appl. Math. 4, (1951). 75–94.

3. **Scattering theory.** Scattering theory is about properties of solutions of the eigenfunction equation $Hu = \lambda^2 u$ where $\lambda \in \mathbb{R}$, u is a function defined on $\mathbb{R}^n$ and H is a differential operator acting on functions on $\mathbb{R}^n$, usually closely related to the Laplacian $\Delta = -\sum_{i=1}^n \partial_{x_i}^2$. For example, H might be $\Delta + V$, where V is the operator of multiplication by a function $V(x)$, representing potential energy (by contrast, Δ represents kinetic energy). There is typically a big solution space to such equations, and a 'basis' for solutions is given by those having some such form in the asymptotic region $|x| \to \infty$ as

$$u(x) = |x|^{-(n-1)/2} \Big(e^{i\lambda|x|} f(x/|x|) + e^{-i\lambda|x|} g(x/|x|) + O(1/|x|) \Big),$$

where f and g are smooth. Moreover, $f(x/|x|, 0)$ is arbitrary, but g is determined by f. In fact, there is a well-defined map $S(\lambda)$ acting on functions defined on the 'sphere at infinity' that maps f to g. This map $S(\lambda)$ is called the scattering matrix (in spite of the fact that it is not a matrix) and characterizes the large-distance effect of the potential function V.

There is a lot known about the scattering matrix, and still plenty of things still to be understood. For example, the high energy limit of the scattering matrix is still not very well understood.

R. B. Melrose, Geometric scattering theory, Stanford Lectures, Cambridge University Press, 1995.

4. **Wavelets.** The idea of wavelets is to introduce useful bases for the Hilbert space $L^2(\mathbb{R}^n)$. 'Useful' means useful for harmonic analysis or PDE. The traditional method for solving such problems is to use the Fourier transform. While this is extremely powerful in many situations, it does have some disadvantages, mostly due to its nonlocality. That is, a particular function f may have localized support, but it is not easy to see this in terms of its Fourier transform $\hat{f}$. Wavelets deal effectively with this issue. There are many applications of wavelets both in theory (they simplify the proofs of many problems) and practise (e.g. compressing information, such as jpg files).

Yves Meyer, Wavelets and Operators, Cambridge studies in advanced mathematics, Cambridge University Press 1992.

5. **Computational problems.** The problem of numerically computing eigenvalues and eigenfunctions of the Laplacian, with Dirichlet (zero) boundary conditions, on a plane domain, is computationally intensive and there is a lot of theory behind finding efficient algorithms. Proving convergence rates is likewise an interesting theoretical problem. Recently, Barnett and Barnett-Hassell have shown that the method of particular solutions (MPS), a standard method, is more accurate

by an order of $E^{1/2}$, where E is the eigenvalue, then previously shown. Analyzing the scaling method, which is a more efficient method for finding large blocks of eigenvalues simultaneously, is planned for 2009. There are good projects possible here for those who like to combine theory and computation.

Look at: Perturbative analysis of the Method of Particular Solutions for improved inclusion of high-lying Dirichlet eigenvalues , Alex H. Barnett, accepted, SIAM J. Num. Anal., from the author's website.

http://math.dartmouth.edu/ ahb/pubs.html

6. **Nonlinear heat equation.**

The heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad t \geq 0,\ x \in \mathbb{R}$$

is well understood. Given reasonable initial data $u(0, x)$, the solution exists for all time and is very well-behaved. If a nonlinear term is added,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u^p,$$

then much more interesting things can happen. For example, for some values of p, the solution may 'blow up' (ie, become infinite) at a finite time. The way in which this happens can be analysed, and the asymptotics of the solution near the blow up time specified quite precisely.

This project would involve learning about the linear and nonlinear heat equations, understanding the literature about blowup, and analysing some interesting examples in detail — possibly involving new forms of blow up that have yet been fully described.

Reference: Herrero, M. A.; Velzquez, J. J. L. Some results on blow up for semilinear parabolic problems. Degenerate diffusions (Minneapolis, MN, 1991), 105–125, IMA Vol. Math. Appl., 47, Springer, New York, 1993.